International Journal of Network Dynamics and Intelligence

Article

Fault-tolerant formation consensus control for time-varying multi-agent systems with stochastic communication protocol

Chunyu Li, Yifan Liu, Ming Gao, and Li Sheng*

College of Control Science and Engineering, China University of Petroleum (East China), Qingdao 266580, China
* Correspondence: E-mail: shengli@upc.edu.cn

Received: 4 September 2023
Accepted: 8 November 2023
Published: 26 March 2024

Abstract: This paper is concerned with the problem of fault-tolerant formation consensus control for linear time-varying (LTV) multi-agent systems (MASs) with stochastic communication protocol (SCP). The SCP is introduced to schedule the signal transmission, and only one neighbouring agent is allowed to transmit data at one instant. The purpose of this work is to design a fault-tolerant controller for each agent, so that, for all probabilistic scheduling behaviors, MASs can achieve the H_∞ formation consensus performance. The state and fault are augmented into a new vector, meanwhile, each agent system is written as a singular one and a state observer is designed. By utilizing the estimated information of states and faults, the designed time-varying compensation term can reduce the impacts of unknown external disturbances and faults. Then, a sufficient condition is obtained to guarantee the H_∞ performance constraint over the finite horizon for closed-loop systems. The parameters of observers and controllers are derived by solving coupled backward recursive Riccati difference equations. Finally, a numerical example is given to validate the effectiveness of the proposed fault-tolerant control scheme.

Keywords: fault-tolerant formation consensus control; multi-agent systems; backward recursive Riccati difference equations; stochastic communication protocol

1. Introduction

Formation control is one of the core problems in the research field of MASs. In many tasks, multi-agents need to maintain a predetermined geometrical shape, that is, formation, with each other in the process of moving toward a specific target or direction, such as effective search, patrol and exploration [1–4]. MASs have the characteristics of distributed composition and parallel execution of tasks, so multi-agent formation offers better fault tolerance and mission efficiency than a single agent and can acquire surrounding environment information as well as save fuel or energy in some vehicle systems, such as unmanned aerial vehicles (UAVs) and autonomous underwater vehicles (AUVs) [5,6]. In addition, formation control has a good prospect in industry, military and aviation [7], thus it has exerted much fascination on control communities.

Similar to sensor networks, the structure of MASs is more fragile than general centralized systems [8,9], and coupled with the harsh working environment, agents are easily to prone to the faults or cyberattacks [10], which may propagate to the healthy individuals through the special communication topology [11]. Therefore, to ensure the reliability and safety of MASs, it is essential that a robust and reliable fault-tolerant control (FTC) system completes its operation within an acceptable time window after fault occurrence in the presence of disturbances in the system [12]. The overall goal of FTC systems is to accommodate faults in the system components during operation and maintain stability with little or acceptable degradation in the performance levels [13]. Fault-tolerant formation control is usually considered as a special case of FTC, which is mainly divided into two categories characterized by whether there exists the fault detection and isolation (FDI) module. The active FTC system has its advantages in dealing with all kinds of faults, and the system can achieve the best FTC performance by reconstructing the controller [14]. The passive FTC system draws on the idea of robust control, and the designed controller has strong robustness for the fault of a certain severity [15].

By now, the fault-tolerant formation control over MASs has been widely studied and a rich body of algorithms have been proposed, where the observer approach has been extensively applied in the problem of FTC [16–22]. On the basis of the estimated state, fault and uncertainty information, fault-tolerant controllers were developed to achieve the desired formation objectives. For instance, in [16], two kinds of distributed finite-time observers were presented for each follower, thus eliminating the assumption that the leader's information is derived by all agents directly. The problem of robust relative motion control in a multirobotic system was solved by sliding mode control technique in [17], where the parameters of fault were estimated using a residual-based synchronous fault-detection scheme. In [18], by virtue of introducing fault estimators, the developed controller was effective to compensate for actuator faults, sensor faults and unknown nonlinearity simultaneously. An adaptive fault-tolerant H_∞ output feedback control scheme was developed based on unknown input observers in [19,20]. Moreover, the fault-tolerant time-varying formation control problem has caused considerable attention from a variety of communities and a lot of distributed algorithms have been developed in [23–26]. However, only certain types of faults considered in the design stage can be treated by the passive FTC system. In general, from the perspective of performance, active FTC is superior to passive FTC scheme.

Compared with time-invariant MASs, the convergence analysis on the FTC problem for time-varying MASs is more challenging due to the complexity caused by time variance [27]. Consensus problem is a fundamental problem in cooperative control of MASs, to mention a few, the leader-following consensus for time-varying MASs has been adequately discussed in [27–32]. Besides, in [33–36], many distributed algorithms have been developed for consensus problems of nonlinear time-varying MASs models expressed via various representations. However, it should be pointed out that studies on fault-tolerant formation control problem for time-varying MASs has not been properly studied so far.

In a typical network with limited bandwidth, multiple simultaneous transmissions on the network will lead to inevitable data conflicts. One of the effective ways to prevent data conflicts is to arrange the transmission of signals according to specific communication protocols. Communication protocols are extensively applied in MASs including SCP [37,38], try-once-discard (TOD) protocol [39], the round-robin (RR) protocol [40,41], and the random-access (RA), etc. Moreover, the use of communication protocols makes the system information incomplete [42], and hence the development of new control algorithms is of great significance to deal with the incomplete information.

Invoked by the above literature review, in this paper, we aim to deal with the problem of distributed fault-tolerant formation consensus control for a class of LTV MASs. This problem is by no means trivial due to the following identified technical challenges:

1) For a LTV MAS with unknown disturbances and faults, the faults will propagate along the communication topology, resulting in a significant reduction in formation consensus performance. How to design a FTC scheme to improve the system robustness to disturbances and faults?

2) For a LTV MAS under SCP scheduling, how to deal with the difficulty in the analysis of fault-tolerant formation consensus control problem caused by SCP scheduling?

3) How to obtain the state and fault information of the system simultaneously by the observer technique?

In light of those three questions, the main contributions of this paper are highlighted as follows:

1) A novel fault-tolerant formation consensus control scheme is, for the first time, proposed for a LTV MAS subjected to unknown external disturbances, system and sensor faults, directed topologies as well as SCP.

2) Under the premise of allowing a small sacrifice of formation consensus performance, a time-varying fault-tolerant controller is designed for each agent based on estimation information of observers, which can tolerate system and sensor faults without making any assumptions on faults.

3) A sufficient condition for the existence of fault-tolerant H_∞ formation consensus controllers is given, and a novel coupled backward recursive RDE method is proposed for time-varying systems with the stochastic parameter matrices.

The rest of this article is organized as follows. In Section II, some basic knowledge of graph theory, the discrete LTV MAS as well as the structures of observers and fault-tolerant formation consensus controllers are introduced, and the problem under the consideration is formulated. In Section III, the observer and controller parameters are obtained by solving backward recursive RDEs. Furthermore, a numerical illustrative example is given in Section IV to show the feasibility and effectiveness of the proposed controller design scheme. Finally, the conclusion is drawn in Section V.

Notations. $\mathbb{R}$ is the space of all real numbers. $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ respectively represent the n-dimensional Euclidean space and the set of all $m\times n$ real matrices. A^T denotes the transpose of a matrix A. $\otimes$ denotes the operation of Kronecker product. $\circ$ is the Hadamard product of matrices. I is the identity matrix with appropriate dimensions, I_n denotes the n-dimensional identity matrix, and $\mathbf{0}$ denotes a zero matrix with appropriate dimensions. $\mathbf{1}_n$ denotes an n dimensional column vector with all ones. The superscript $\dagger$ denotes Moore-Penrose pseudo inverse. $\text{diag}\{P_1, P_2, \cdots, P_n\}$ stands for a block-diagonal matrix whose diagonal elements are $P_1, P_2, \cdots, P_n$. The notation $A > B$, where A and B are real symmetric matrices, means that $A - B$ is positive definite. $\text{Prob}\{\cdot\}$ refers to the occurrence probability of the event $\{\cdot\}$. $\mathbb{E}\{x\}$ denotes the mathematical expectation of x. $\|x\|$ represents the Euclidean norm of x, and $\|A\|_F$ denotes the Frobenius norm of A. $\rho(A)$ denotes the spectral radius of the square matrix A. $\vec{\delta}(\cdot)$ represents the Kronecker delta function with $\vec{\delta}(a)=1$ (if $a=0$) and $\vec{\delta}(a)=0$ (otherwise).

2. Problem Formulation and Preliminaries

2.1. Graph Theory

The communication topology among the N agents is described by a weighted directed graph $G=\{V,E,A\}$, where $V=\{v_1,v_2,\cdots,v_N\}$ represents the collection of nodes, and v_i represents the i-th node of the graph G. In the description of MASs, the nodes represent the location of agents. A directed edge (v_i,v_j) means agent i can receive information from agent j, and $E\subseteq V\times V$ means the set of directed edges, where $E=\{(v_i,v_j):v_i,v_j\in V, i\neq j\}$. $A=\left[a_{i,j}\right]\in\mathbb{R}^{N\times N}$ denotes the weighted adjacency matrix, which consists of the non-negative elements $a_{i,j}$ satisfying $a_{i,j}>0\Leftrightarrow(v_i,v_j)\in E$. In this paper, self-edges are not allowed, which means $(v_i,v_i)\notin E$ and $a_{i,i}=0$ for any $v_i\in V$. $N_i=\{v_j\in V:(v_i,v_j)\in E, i\neq j\}$ means the neighborhood of agent v_i. $D=\operatorname{diag}\{d_1,d_2,\ldots,d_N\}$ is on behalf of the degree matrix with $d_i=\sum_{j\in N_i}a_{i,j}$. The Laplacian matrix $L=A-D\in\mathbb{R}^{N\times N}$ consists of the elements $l_{i,j}$, and $l_{i,i}=\sum_{j=1,j\neq i}^{N}a_{i,j}$, $l_{i,j}=-a_{i,j}$ for all $i\neq j$.

2.2. Problem Formulation

Considering a MAS composed of N identical agents, in order to prevent data collisions, we assume that for agent l, only one neighboring agent $i\in N_i$ is allowed to transmit data to agent l at each transmission instant. Let $\varepsilon_l(k)\in N_l$ represent the neighboring agent which gets the opportunity to transmit data to the agent l at time instant k. As discussed in [43], $\varepsilon_l(k)$ could be regarded as a sequence of random variables to represent the scheduling behavior of the SCP. All the random variables $\varepsilon_l(k)(l\in\{1,2,\cdots,N\})$ are mutually independent, and the probability of $\varepsilon_l(k)=i$ is

$$\operatorname{Prob}\{\varepsilon_l(k)=i\}=p_l^i(k),\ l\in\{1,2,\cdots,N\},\ i\in N_l, \tag{1}$$

where $p_l^i(k)\geqslant 0$ represents that agent i transmits data to agent l at time instant k, and there has been a common assumption that $\sum_{i\in N_l}p_l^i(k)=1$, and $p_l^i(k)=0\,(i\notin N_l)$.

Remark 1: As discussed in extensive existing literature, the SCP is generally employed to determine which node has the priority of accessing to the communication networks at each transmission instant, which can effectively reduce the communication burden among sensors [44]. The probabilistic accesses of SCP have been identified as a set of random variables affecting the system performance [45]. For the considered SCP, $\varepsilon_l(k)$ is a stochastic process, which can randomly generate the number from the set N_l (e.g. the neighborhood of agent l) subject to the transition probability determined in (1).

Consider the following discrete LTV MAS defined on the finite horizon ($k\in[0,T-1]$) and composed of N agents, whose topology is a directed graph, and the dynamics of the l-th agent is described by

$$\begin{cases} x_l(k+1)=A(k)x_l(k)+B(k)u_l(k)+E(k)w_l(k)+E_f(k)f_l(k),\\ \quad y_l(k)=C(k)x_l(k)+D(k)v_l(k)+F_f(k)f_l(k),\\ \quad r_l(k)=\bar{V}(k)x_l(k),\ l=1,2,\cdots,N, \end{cases} \tag{2}$$

where $x_l(k)\in\mathbb{R}^{n_x}$, $y_l(k)\in\mathbb{R}^{n_y}$, $u_l(k)\in\mathbb{R}^{n_u}$, $f_l(k)\in\mathbb{R}^{n_f}$ and $r_l(k)\in\mathbb{R}^{n_r}$ represent the system state, the measurement output, the control input, the fault and the control output respectively for the l-th agent. $w_l(k)\in L_2([0,T-1];\mathbb{R}^{n_w})$ and $v_l(k)\in L_2([0,T-1];\mathbb{R}^{n_v})$ are external disturbances, respectively. $w_l(k)$ satisfies $\|w_l(k)\|\leqslant\varpi, l\in\{1,2,\cdots,N\}$, where ϖ is a non-negative constant. $A(k)$, $B(k)$, $E(k)$, $E_f(k)$, $C(k)$, $D(k)$, $F_f(k)$ and $\bar{V}(k)$ are known and time-varying matrices with appropriate dimensions.

To facilitate the analysis of the problem, the following definitions are needed. Define the reference formation as a time-varying matrix $l(k)=\left[l_1^T(k),\cdots,l_N^T(k)\right]^T(l_l(k)\in\mathbb{R}^{n_x})$, which is assumed to be known. Subsequently, define formation error and formation consensus error for agent l as $e_l^f(k)\triangleq x_l(k)-l_l(k)$ and $e_l^c(k)\triangleq e_l^f(k)-\dfrac{1}{N}\sum_{i=1}^{N}e_i^f(k)$, respectively.

The premise of achieving fault-tolerant formation consensus control is to obtain the state and the fault information of the system. For the sake of the subsequent analysis, by defining $\xi_l(k)=\left[x_l^T(k)\quad f_l^T(k)\right]^T$, system (2) can be rewritten as the following singular system:

$$\begin{cases} \bar{E}\xi_l(k+1)=\bar{A}(k)\xi_l(k)+B(k)u_l(k)+E(k)w_l(k),\\ \quad y_l(k)=\bar{C}(k)\xi_l(k)+D(k)v_l(k),\\ \quad r_l(k)=V(k)\xi_l(k), \end{cases} \tag{3}$$

where $\bar{E} \triangleq \begin{bmatrix} I_{n_x} & 0_{n_x \times n_f} \end{bmatrix}$, $\bar{A}(k) \triangleq \begin{bmatrix} A(k) & E_f(k) \end{bmatrix}$, $\bar{C}(k) \triangleq \begin{bmatrix} C(k) & F_f(k) \end{bmatrix}$, $V(k) \triangleq \bar{V}(k)\bar{E}$. Assume that $C(k)$ is row full rank and $(\bar{C}(k), \bar{A}(k))$ is observable.

In this article, the problem of finite-horizon formation consensus control is studied for the LTV MAS under SCP. In the objective system, exogenous disturbances, sensor and system faults are considered simultaneously. To deal with this problem, an observer is designed for each agent to obtain the system state and fault information, and a new distributed active FTC method is developed via adding a compensation term to alleviate the effects of system faults, time-varying parameters and external disturbances on the formation consensus among the agents. The following introduces the definition of achieving H_∞ formation consensus for MAS (2).

Definition 1: Let a disturbance attenuation level $\gamma_1 > 0$ and a positive definite matrix W_1 be given. Considering MAS (2) with a connected directed topology, if

$$\mathbb{E}\left\{ \sum_{l=1}^{N} \sum_{k=0}^{T-1} \left(\left\| \bar{e}_l^c(k) \right\|^2 - \gamma_1^2 \left\| \bar{\delta}_l(k) \right\|^2 \right) - \gamma_1^2 (e^c(0))^T (I_N \otimes W_1) e^c(0) \right\} < 0 \tag{4}$$

holds, then the MAS is said to satisfy the H_∞ consensus performance constraint over the finite horizon ($k \in [0, T-1]$), where $\bar{e}_l^c(k) \triangleq \bar{V}(k) e_l^c(k)$, and $\bar{\delta}_l(k)$ represents the influence of faults, estimation errors and external disturbances on the formation consensus error system, and its definition will be given later.

2.3. State Observers Design

Assumption 1: At each transmission instant k in the finite horizon $[0, T-1]$, there exists $\{X(k+1)\}_{0 \leqslant k \leqslant T-1}$ that would make rank $\left(\begin{bmatrix} \bar{C}^T(k+1) & I - \bar{E}^T X^T(k+1) \end{bmatrix} \right) = \text{rank} \left(\bar{C}(k+1) \right)$ holds.

In order to estimate the faults and system states simultaneously, a decentralized state observer is designed for system (3):

$$\begin{cases} z_l(k+1) = M(k) z_l(k) + T(k) u_l(k) + G(k) y_l(k) - Q(k) (y_l(k) - \hat{y}_l(k)), \\ \hat{\xi}_l(k+1) = z_l(k+1) + R(k+1) y_l(k+1), \end{cases} \tag{5}$$

where $z_l(k) \in \mathbb{R}^{n_x + n_f}$ and $\hat{\xi}_l(k)$ respectively denote the observer state and the estimation of augmented vector $\xi_l(k)$. $\{Q(k), M(k), T(k), G(k), R(k)\}_{0 \leqslant k \leqslant T-1}$ are the observer parameters that need to be computed, and there exists a sequence of matrices $\{X(k+1)\}_{0 \leqslant k \leqslant T-1}$ satisfying the following time-varying equations:

$$X(k+1)\bar{E} + R(k+1)\bar{C}(k+1) = I_{n_x + n_f}, \tag{6}$$

$$X(k+1) B(k) = T(k), \tag{7}$$

$$X(k+1) \bar{A}(k) = M(k), \tag{8}$$

$$M(k) R(k) = G(k), \tag{9}$$

where Assumption 1 guarantees that equation (6) must have a non-zero solution $\{R(k+1)\}_{0 \leqslant k \leqslant T-1}$.

In this paper, the idea of active fault-tolerant scheme is employed to deal with formation consensus control problem, and the FTC scheme is shown in Figure 1. By utilizing the state and fault information obtained by the estimator, the fault-tolerant formation consensus controllers would be construct for the MAS to compensate for the influence of faults and exogenous disturbances.

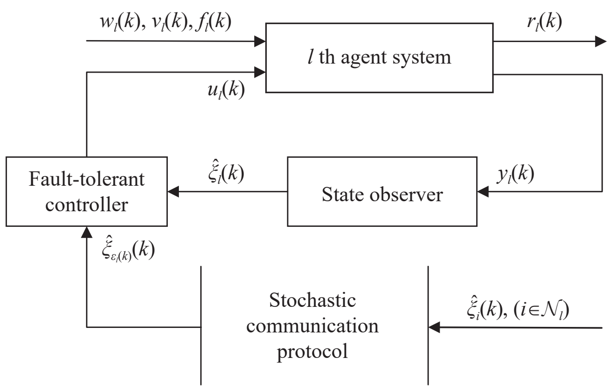

Figure 1. Structure of fault-tolerant distributed formation consensus control scheme.

2.4. Formation Consensus Controllers Design

The core idea of the fault-tolerant controller design is to attenuate the influence of external disturbances and faults by adding the compensation term to the formation consensus controller. Under the SCP scheduling, the fault-tolerant formation consensus controller is designed as follows:

$$\begin{aligned} u_l(k) &= K(k)\left(\bar{E}\hat{\xi}_{\varepsilon_l(k)}(k) - l_{\varepsilon_l(k)}(k) - \bar{E}\hat{\xi}_l(k) + l_l(k)\right) - \bar{K}(k)\left(\tilde{l}_l(k) + E_f(k)\tilde{E}\hat{\xi}_l(k)\right) \\ &= K(k)\sum_{i\in N_l} a_{l,i}\lambda_i^l(k)\left(\bar{E}\hat{\xi}_{\varepsilon_l(k)}(k) - l_{\varepsilon_l(k)}(k) - \bar{E}\hat{\xi}_l(k) + l_l(k)\right) - \bar{K}(k)\left(\tilde{l}_l(k) + E_f(k)\tilde{E}\hat{\xi}_l(k)\right), \end{aligned} \tag{10}$$

where $\{K(k), \bar{K}(k)\}_{0\leqslant k\leqslant T-1} \in \mathbb{R}^{n_u\times n_x}$ are the gain matrices to be determined, $\lambda_i^l(k) = \vec{\delta}(i - \varepsilon_l(k)), (i \in N_l)$, $\tilde{l}_l(k) \triangleq A(k)l_l(k) - l_l(k+1)$, $\tilde{E} \triangleq \begin{bmatrix} 0_{n_f\times n_x} & I_{n_f} \end{bmatrix}$, and $a_{l,i}$ is the element in the l-th row and the i-th column of the weighted adjacency matrix A.

Defining estimation errors of augmented states $e_l^{\xi}(k) \triangleq \xi_l(k) - \hat{\xi}_l(k)$, other variables

$$\begin{aligned} e^{\xi}(k) &= \left[e_1^{\xi}(k), \cdots, e_N^{\xi}(k)\right]^T, u(k) = [u_1(k), \cdots, u_N(k)]^T, e^f(k) = \left[e_1^f(k), \cdots, e_N^f(k)\right]^T, \\ \tilde{l}(k) &= \left[\tilde{l}_1(k), \cdots, \tilde{l}_N(k)\right]^T, \hat{\xi}(k) = \left[\hat{\xi}_1(k), \cdots, \hat{\xi}_N(k)\right]^T, \varepsilon(k) = [\varepsilon_1(k), \cdots, \varepsilon_N(k)]^T, \end{aligned}$$

yields the following compact form of fault-tolerant controller (10):

$$\begin{aligned} u(k) = &\left(I_N \otimes K(k)\right)\left(\vec{L}(\varepsilon(k)) \otimes I_{n_x}\right)\left(e^f(k) - \left(I_N \otimes \bar{E}\right)e^{\xi}(k)\right) - \left(I_N \otimes \bar{K}(k)\right)\tilde{l}(k) \\ &- \left(I_N \otimes \bar{K}(k)E_f(k)\tilde{E}\right)\hat{\xi}(k) \end{aligned} \tag{11}$$

where

$$\begin{aligned} &\vec{L}(\varepsilon(k)) \triangleq A \circ \vec{\Lambda}(k) - \vec{D}(k), \vec{\Lambda}(k) \triangleq \left[\lambda_j^i(k)\right]_{N\times N}, \vec{D}(k) \triangleq \operatorname{diag}\left\{\vec{d}_1(k), \cdots, \vec{d}_N(k)\right\}, \\ &\vec{d}_i(k) \triangleq \sum\nolimits_{j\in N_i} a_{i,j}\lambda_j^i(k). \end{aligned}$$

Remark 2: Obviously, $\vec{L}(\varepsilon(k))$ is a stochastic parameter matrix whose value depends on the sequences $\varepsilon_l(k) (l \in \{1, 2, \cdots, N\})$. It can be seen that $\vec{d}_i(k)$ indicates the i-th row-sum of $\vec{L}(\varepsilon(k))$. Bring the SCP in mind, if $a_{i,j}\lambda_i^l(k)$ is regarded as the element of the new adjacency matrix, then $A \circ \vec{\Lambda}(k)$ acts as the adjacency matrix, and $\vec{D}(k)$ indicates the degree matrix in this case.

To facilitate the subsequent analysis, a relationship between $\varepsilon_l(k)$ and one random sequence $h(k)$ is developed by mapping technique.

Lemma 1: The random variable sequences $\varepsilon_l(k)$ $(l \in 1, 2, \cdots, N, k = 0, 1, \cdots, T)$ representing the scheduling behavior of SCP can be mapped to sequence $h(k) \in R \triangleq \left\{1, 2, \cdots, N^N, k = 0, 1, \cdots, T\right\}$:

$$h(k) = H(\varepsilon(k)) = \sum_{l=1}^{N} N^{l-1}\left(\varepsilon_l(k) - 1\right) + 1. \tag{12}$$

Moreover, if $h(k)$ is given, then the value of $\varepsilon_l(k)$ can be obtained by $\rho_l(h(k)) (l = 1, 2, \cdots, N)$:

$$\varepsilon_l(k) = \rho_l(h(k)) = \bmod\left(\left\lfloor \frac{h(k)}{N^{l-1}} \right\rfloor, N\right) + 1. \tag{13}$$

Proof: By noting (12), it readily follows that, for sequence $h(k)$ calculated by (12), $h(k) \in R$ holds. Below we continue the proof on that $\varepsilon_l(k)$ can be acquired by (13). Since $\varepsilon_l(k) - 1 < N$ holds for all $l \in \{1, 2, \cdots, N\}$, for any given $h(k) \in R$, it follows

$$\begin{aligned} \rho_l(h(k)) &= \bmod\left(\left\lfloor \frac{h(k)}{N^{l-1}} \right\rfloor, N\right) + 1 \\ &= \bmod\left(\sum_{j=0}^{N-l} N^{N-j-l}\left(\varepsilon_{N-j}(k) - 1\right), N\right) + 1 \\ &= \varepsilon_l(k). \end{aligned}$$

It can be concluded that there is a one-to-one correspondence between the variable $h(k)$ and the vector $\varepsilon(k)$. Hence, the peculiarity of $h(k)$ is determined by the random sequences $\varepsilon_l(k) (l \in 1, 2, \cdots, N)$.

Lemma 2: For any $k \in [0, T-1]$, the occurrence probability of $h(k) = i \in R$ is

$$\begin{aligned}\bar{p}_i(k) &= \text{Prob}\{h(k)=i\} \\ &= \prod_{l=1}^{N} \text{Prob}\{\varepsilon_l(k)=\rho_l(i)\} \\ &= \prod_{l=1}^{N} p_l^{\rho_l(i)}(k).\end{aligned} \tag{14}$$

Proof: Because $\varepsilon_l(k)=\rho_l(h(k))$ holds for all $l\in\{1,2,\cdots,N\}$, and $\varepsilon_l(k)$ is a random variables with independent distribution according to the character of SCP, it yields (14) by noticing (1), where continued multiplication indicates intersection, that is $\varepsilon_l(k)=\rho_l(i)$ holds simultaneously for all $l\in\{1,2,\cdots,N\}$.

2.5. Closed-Loop System

The following lemma is presented before moving to next part.

Lemma 3: [46] For arbitrary matrices $A\in\mathbb{R}^{n\times n}$ and $B\in\mathbb{R}^{n\times n}$, the row-sums of $A\circ B$ are the diagonal elements of AB^T, i.e.,

$$\sum_{j=1}^{N}(A\circ B)_{i,j} = \left(AB^T\right)_{i,i}.$$

For the sake of presentation, let us introduce some notations:

$$\xi(k)=\left[\xi_1(k),\cdots,\xi_N(k)\right]^T, y(k)=\left[y_1(k),\cdots,y_N(k)\right]^T, \hat{y}(k)=\left[\hat{y}_1(k),\cdots,\hat{y}_N(k)\right]^T,$$
$$w(k)=[w_1(k),\cdots,w_N(k)]^T, v(k)=[v_1(k),\cdots,v_N(k)]^T, z(k)=[z_1(k),\cdots,z_N(k)]^T.$$

Then, the MAS (3) and the observers (5) can be respectively written in the following compact forms:

$$\begin{cases}\left(I_N\otimes\bar{E}\right)\xi(k+1)=(I_N\otimes\bar{A}(k))\xi(k)+(I_N\otimes B(k))u(k)+(I_N\otimes E(k))w(k),\\ y(k)=(I_N\otimes\bar{C}(k))\xi(k)+(I_N\otimes D(k))v(k),\\ r(k)=V(k)\xi(k),\end{cases} \tag{15}$$

and

$$\begin{cases}z(k+1)=(I_N\otimes M(k))z(k)+(I_N\otimes T(k))u(k)+(I_N\otimes G(k))y(k)-(I_N\otimes Q(k))(y(k)-\hat{y}(k)),\\ \hat{\xi}(k+1)=z(k+1)+(I_N\otimes R(k+1))y(k+1),\end{cases} \tag{16}$$

where $V(k)\triangleq I_N\otimes V(k)$.

Subsequently, the estimation error system can be obtained by (10)

$$\begin{aligned}e^{\xi}(k+1) &= \xi(k+1)-\hat{\xi}(k+1)\\ &= \xi(k+1)-z(k+1)-(I_N\otimes R(k+1))y(k+1)\\ &= \xi(k+1)-(I_N\otimes M(k))z(k)-(I_N\otimes T(k))u(k)-(I_N\otimes G(k))y(k)+(I_N\otimes Q(k))(y(k)-\hat{y}(k))\\ &\quad-\left(I_N\otimes R(k+1)\bar{C}(k+1)\right)\xi(k+1)-(I_N\otimes R(k+1)D(k+1))v(k+1)\\ &= \left(I_N\otimes\left(I-R(k+1)\bar{C}(k+1)\right)\right)\xi(k+1)-(I_N\otimes M(k))z(k)-(I_N\otimes T(k))u(k)\\ &\quad-(I_N\otimes G(k))y(k)+(I_N\otimes Q(k))(y(k)-\hat{y}(k))-(I_N\otimes R(k+1)D(k+1))v(k+1).\end{aligned} \tag{17}$$

Substituting (6) into (17) and considering (3), it follows

$$\begin{aligned}e^{\xi}(k+1) = {}&\left(I_N\otimes X(k+1)\bar{A}(k)\right)\xi(k)+(I_N\otimes X(k+1)B(k))u(k)+(I_N\otimes X(k+1)E(k))w(k)\\ &-(I_N\otimes M(k))z(k)-(I_N\otimes T(k))w(k)-(I_N\otimes G(k))y(k)+(I_N\otimes Q(k))(y(k)-\hat{y}(k))\\ &-(I_N\otimes R(k+1)D(k+1))v(k+1).\end{aligned} \tag{18}$$

Implementing (5), (7) and (8) into (18), we have

$$\begin{aligned}e^{\xi}(k+1) &= (I_N\otimes M(k))e^{\xi}(k)+(I_N\otimes(M(k)R(k)-G(k)))y(k)-(I_N\otimes R(k+1)D(k+1))v(k+1)\\ &\quad+(I_N\otimes Q(k))\left(\left(I_N\otimes\bar{C}(k)\right)e^{\xi}(k)+(I_N\otimes D(k))v(k)\right)+(I_N\otimes X(k+1)E(k))w(k)\\ &= \left(I_N\otimes\left(M(k)+Q(k)\bar{C}(k)\right)\right)e^{\xi}(k)+(I_N\otimes Q(k)D(k))v(k)-(I_N\otimes R(k+1)D(k+1))v(k+1)\\ &\quad+(I_N\otimes X(k+1)E(k))w(k).\end{aligned} \tag{19}$$

By defining variables $\delta(k)=\left[w^T(k)\quad v^T(k)\quad v^T(k+1)\right]^T$, $L(k)=I_N\otimes\left(M(k)+Q(k)\bar{C}(k)\right)$, and $Y(k)=[I_N\otimes X(k+1)E(k)\quad I_N\otimes Q(k)D(k)\quad -I_N\otimes R(k+1)D(k+1)]$, the following estimation error system is derived from

(19):

$$e^{\xi}(k+1) = L(k)e^{\xi}(k) + Y(k)\delta(k). \tag{20}$$

One of the research objectives of this paper is to design the observer parameters $\{Q(k)\}_{0\leqslant k\leqslant T-1}$ so that the estimation error $\bar{e}^{\xi}(k) \triangleq r(k) - \hat{r}(k)$ of the controlled output $r(k)$ satisfies the H_∞ performance constraint defined below.

Definition 2: Let a disturbance attenuation level $\gamma_2 > 0$ and a positive definite matrix W_2 be given. Considering LTV MAS (2), for any disturbance sequences $\{w(k), v(k)\}_{0\leqslant k\leqslant T-1}$, if the following inequality is satisfied, then the estimation error system (20) satisfies the H_∞ consensus performance over the finite horizon ($k \in [0, T-1]$):

$$J_1 = \sum_{k=0}^{T-1} \left(\left\|\bar{e}^{\xi}(k)\right\|^2 - \gamma_2^2 \|\delta(k)\|^2 \right) - \gamma_2^2 \left(e^{\xi}(0)\right)^T (I_N \otimes W_2) e^{\xi}(0) < 0,\ \forall \left(\delta(k), e^{\xi}(0)\right) \neq 0, \tag{21}$$

where it can be known from the definition of $\bar{e}^{\xi}(k)$ that $\bar{e}^{\xi}(k) = V(k)e^{\xi}(k)$.

By denoting formation consensus error $e^c(k) = \left[e_1^c(k), \ldots, e_N^c(k)\right]^T$, it can be inferred that

$$e^c(k) = \left(K \otimes I_{n_x}\right) e^f(k),$$

where $K \triangleq I_N - \frac{1}{N} 1_N 1_N^T$. Subsequently, it readily follows that the formation consensus error system is

$$\begin{aligned} e^c(k+1) &= \left(K \otimes I_{n_x}\right)(x(k+1) - l(k+1)) \\ &= (K \otimes A(k))\, x(k) + (K \otimes B(k))\, u(k) + (K \otimes E(k))\, w(k) + \left(K \otimes E_f(k)\right) f(k) \\ &\quad - \left(K \otimes I_{n_x}\right) l(k+1). \end{aligned} \tag{22}$$

Substituting the controller (11) into (22) yields

$$\begin{aligned} e^c(k+1) &= (K \otimes A(k))\, x(k) + \left(K\hat{\Lambda}(h(k)) \otimes B(k)K(k)\right) \left(e^f(k) - \left(I_N \otimes \bar{E}\right) e^{\xi}(k)\right) - \left(K \otimes B(k)\bar{K}(k)\right) \tilde{l}(k) \\ &\quad - \left(K \otimes B(k)\bar{K}(k)E_f(k)\tilde{E}\right) \hat{\xi}(k) + (I_N \otimes E(k))\, w(k) + \left(K \otimes E_f(k)\tilde{E}\right) \xi(k) - \left(K \otimes I_{n_x}\right) l(k+1), \end{aligned} \tag{23}$$

where $\hat{\Lambda}(h(k)) = \vec{L}(\varepsilon(k))$.

Then, (23) can be further written into

$$\begin{aligned} e^c(k+1) &= \left(K \otimes A(k) + K\hat{\Lambda}(h(k)) \otimes B(k)K(k)\right) e^f(k) - \left(K\hat{\Lambda}(h(k)) \otimes B(k)K(k)\bar{E}\right) e^{\xi}(k) \\ &\quad + (I_N \otimes E(k))\, w(k) + (K \otimes A(k))\, l(k) - \left(K \otimes I_{n_x}\right) l(k+1) - \left(K \otimes B(k)\bar{K}(k)\right) \tilde{l}(k) \\ &\quad + \left(K \otimes E_f(k)\tilde{E}\right) \xi(k) - \left(K \otimes B(k)\bar{K}(k)E_f(k)\tilde{E}\right) \hat{\xi}(k) \\ &= \left(K \otimes A(k) + K\hat{\Lambda}(h(k)) \otimes B(k)K(k)\right) e^f(k) + \left(K \otimes E_f(k)\tilde{E} - K\hat{\Lambda}(h(k)) \otimes B(k)K(k)\bar{E}\right) e^{\xi}(k) \\ &\quad + (I_N \otimes E(k))\, w(k) + \left(K \otimes I_{n_x} - K \otimes B(k)\bar{K}(k)\right) \left(\tilde{l}(k) + \left(I_N \otimes E_f(k)\tilde{E}\right) \hat{\xi}(k)\right). \end{aligned} \tag{24}$$

In terms of Lemma 3, the following equation can be obtained:

$$\sum_{j=1}^{N} \left(\hat{\Lambda}(h(k))\right)_{i,j} = \sum_{j=1}^{N} \left(A \circ \vec{\Lambda}(k)\right)_{i,j} - \vec{d}_i(k) = \left(A\vec{\Lambda}^T(k)\right)_{i,i} - \sum_{j\in N_l} a_{i,j}\lambda_j^i(k) = 0, \tag{25}$$

which means all the row-sums of $\hat{\Lambda}(h(k))$ equal to 0. Therefore, it can be proved that the equation $\hat{\Lambda}(h(k))$ $K = \hat{\Lambda}(h(k))$ holds.

On the basis of (24), it can be inferred that

$$\begin{aligned} e^c(k+1) &= \left(I_N \otimes A(k) + K\hat{\Lambda}(h(k)) \otimes B(k)K(k)\right) e^c(k) + \left(K \otimes I_{n_x} - K \otimes B(k)\bar{K}(k)\right) \tilde{w}(k) \\ &\quad + \left(K \otimes E_f(k)\tilde{E} - K\hat{\Lambda}(h(k)) \otimes B(k)K(k)\bar{E}\right) e^{\xi}(k) + (I_N \otimes E(k))\, w(k), \end{aligned} \tag{26}$$

where $\tilde{w}(k) \triangleq \tilde{l}(k) + \left(I_N \otimes E_f(k)\tilde{E}\right) \hat{\xi}(k)$.

Subject to the distribution of $h(k)$ given by Lemma 2, one obtains

$$\bar{\Lambda}(k) \triangleq \mathbb{E}\left\{\hat{\Lambda}(h(k))\right\} = \sum_{i\in R} \bar{p}_i(k)\hat{\Lambda}(i). \tag{27}$$

Define $\tilde{\Lambda}(h(k)) \triangleq \hat{\Lambda}(h(k)) - \bar{\Lambda}(k)$, then (26) can also be equivalently expressed as

$$e^c(k+1) = \left(\tilde{A}(k) + \tilde{B}_{h(k)}(k)\right) e^c(k) + \left(\bar{H}(k) + \bar{M}_{h(k)}(k)\right) \bar{\eta}(k) + \bar{N}(k)\tilde{w}(k), \tag{28}$$

where

$$\tilde{A}(k) \triangleq I_N \otimes A(k) + K\bar{\Lambda}(k) \otimes B(k)K(k), \tilde{B}_{h(k)}(k) \triangleq K\tilde{\Lambda}(h(k)) \otimes B(k)K(k),$$
$$\bar{H}(k) \triangleq \begin{bmatrix} I_N \otimes E_f(k)\tilde{E} - K\bar{\Lambda}(k) \otimes B(k)K(k)\bar{E} & I_N \otimes E(k) \end{bmatrix}, \bar{N}(k) \triangleq K \otimes I_{n_x} - K \otimes B(k)\bar{K}(k),$$
$$\bar{M}_{h(k)}(k) \triangleq \begin{bmatrix} -K\tilde{\Lambda}(h(k)) \otimes B(k)K(k)\bar{E} & 0_{Nn_x \times Nn_w} \end{bmatrix}, \bar{\eta}(k) \triangleq \begin{bmatrix} \left(e^{\xi}(k)\right)^T & w^T(k) \end{bmatrix}^T.$$

Summarizing the aforementioned discussion, in this paper, we aim to deal with the problem of designing the controller gains $K(k)$ and $\bar{K}(k)$ which can ensure that the H_∞ performance is achieved for formation consensus error $e^c(k)$. The core idea is to attenuate the effect of unknown factors including estimation errors, faults, external disturbances and changing formation control targets on achieving formation consensus of MASs. Recalling the form of formation consensus error system (28), we treat $e^{\xi}(k)$, $w(k)$ and $\tilde{w}(k)$ as unknown disturbances, and then give the following definition:

Definition 3: For all possible realizations of the random sequence $h(k)$, if inequality

$$J_2 = \mathbb{E}\left\{ \sum_{k=0}^{T-1} \left(\|\bar{e}^c(k)\|^2 - \gamma_1^2 \left\|\bar{\delta}(k)\right\|^2 \right) - \gamma_1^2 (e^c(0))^T (I_N \otimes W_1) e^c(0) \right\} < 0, \ \forall \left(\bar{\delta}(k), e(0)\right) \neq 0, \tag{29}$$

holds over the finite horizon $[0, T-1]$, then H_∞ formation consensus is achieved for time-varying MAS (2), where $\bar{e}^c(k) = \begin{bmatrix} \bar{e}_1^c(k), \cdots, \bar{e}_N^c(k) \end{bmatrix}^T$, $\bar{\delta}(k) = \begin{bmatrix} \bar{\delta}_1(k), \cdots, \bar{\delta}_N(k) \end{bmatrix}^T$, $\bar{\delta}_l(k) \triangleq \bar{\eta}(k) - \tilde{w}(k)$. In virtue of the definition of $\bar{e}_l^c(k)$ in (4), we can get that $\bar{e}^c(k) = \hat{V}(k)e^c(k)$, where $\hat{V}(k) \triangleq I_N \otimes \bar{V}(k)$.

3. Main Results

In this section, the design methods of observer and controller parameters are proposed.

3.1. Observer Parameters Design

Lemma 4: [47] Let Υ, Γ, and Σ be known nonzero matrices with appropriate dimensions. The solution Θ to $\arg\min_{\Theta} \|\Upsilon\Theta\Sigma - \Gamma\|_F$ is $\Upsilon^{\dagger}\Gamma\Sigma^{\dagger}$.

Theorem 1: Consider time-varying MAS (3) with the decentralized state observer (5), and let the disturbance attenuation level γ_2 and the positive definite matrix $W_2 = W_2{}^T > 0$ be given. Then, for any disturbance sequence $\{w(k), v(k)\}_{0 \leqslant k \leqslant T-1}$, the augmented state estimation error system (20) satisfies the H_∞ performance defined in (21) if there exist solutions $\left\{Q(k), \bar{R}(k)\right\}_{0 \leqslant k \leqslant T-1}$ (with condition $\bar{R}(T) = 0$) satisfying the recursive RDE:

$$\bar{R}(k) = V^T(k)V(k) + L^T(k)\bar{R}(k+1)L(k) + L^T(k)\bar{R}(k+1)Y(k)\Delta_1^{-1}(k)Y^T(k)\bar{R}(k+1)L(k), \tag{30}$$

subject to

$$\bar{R}(0) < \gamma_2^2 (I_N \otimes W_2), \tag{31}$$

$$\Delta_1(k) \triangleq \gamma_2^2 I_{N(2n_v+n_w)} - Y^T(k)\bar{R}(k+1)Y(k) > 0. \tag{32}$$

Proof: Define $U_{\xi}(k) = \left(e^{\xi}(k)\right)^T \bar{R}(k) e^{\xi}(k)$, then defining $\bar{\Psi}_1(k)$ and considering (20) yields

$$\begin{aligned}
\bar{\Psi}_1(k) &\triangleq U_{\xi}(k+1) - U_{\xi}(k) \\
&= \left(L(k)e^{\xi}(k) + Y(k)\delta(k)\right)^T \bar{R}(k+1) \left(L(k)e^{\xi}(k) + Y(k)\delta(k)\right) - \left(e^{\xi}(k)\right)^T \bar{R}(k) e^{\xi}(k) \\
&= \left(e^{\xi}(k)\right)^T \left(L^T(k)\bar{R}(k+1)L(k) - \bar{R}(k)\right) e^{\xi}(k) + 2\left(e^{\xi}(k)\right)^T L^T(k)\bar{R}(k+1)Y(k)\delta(k) \\
&\quad + \delta^T(k)Y^T(k)\bar{R}(k+1)Y(k)\delta(k).
\end{aligned} \tag{33}$$

Adding the zero term $\left\|\bar{e}^{\xi}(k)\right\|^2 - \gamma_2^2\|\delta(k)\|^2 - \left(\left\|\bar{e}^{\xi}(k)\right\|^2 - \gamma_2^2\|\delta(k)\|^2\right)$ to the right side of (33), one obtains

$$\begin{aligned}
\bar{\Psi}_1(k) = {} & \left(e^{\xi}(k)\right)^T \left(L^T(k)\bar{R}(k+1)L(k) - \bar{R}(k) + V^T(k)V(k)\right) e^{\xi}(k) + 2\left(e^{\xi}(k)\right)^T L^T(k)\bar{R}(k+1)Y(k)\delta(k) \\
& - \delta^T(k) \left(\gamma_2^2 I - Y^T(k)\bar{R}(k+1)Y(k)\right) \delta(k) - \left(\left\|\bar{e}^{\xi}(k)\right\|^2 - \gamma_2^2\|\delta(k)\|^2\right).
\end{aligned} \tag{34}$$

By applying the completing squares method, (34) can be converted into the following form:

$$\begin{aligned}
\bar{\Psi}_1(k) = {} & \left(e^{\xi}(k)\right)^T \left(L^T(k)\bar{R}(k+1)L(k) - \bar{R}(k) + V^T(k)V(k)\right) e^{\xi}(k) + \tilde{\delta}^T(k)\Delta_1(k)\tilde{\delta}(k) \\
& - \left(\delta(k) - \tilde{\delta}(k)\right)^T \Delta_1(k) \left(\delta(k) - \tilde{\delta}(k)\right) - \left(\left\|\bar{e}^{\xi}(k)\right\|^2 - \gamma_2^2\|\delta(k)\|^2\right),
\end{aligned} \tag{35}$$

where $\tilde{\delta}(k) \triangleq \Delta_1^{-1}(k)Y^T(k)\bar{R}(k+1)L(k)e^{\xi}(k)$.

Substituting (30) into (35) and summing up it from 0 to T with respect to k yield

$$
\begin{aligned}
&\left(e^{\xi}(T)\right)^T \bar{R}(T)e^{\xi}(T)-\left(e^{\xi}(0)\right)^T \bar{R}(0)e^{\xi}(0) \\
&=-\sum_{k=0}^{T-1}\left(\delta(k)-\tilde{\delta}(k)\right)^T \Delta_1(k)\left(\delta(k)-\tilde{\delta}(k)\right)-\sum_{k=0}^{T-1}\left(\left\|\bar{e}^{\xi}(k)\right\|^2-\gamma_2^2\|\delta(k)\|^2\right).
\end{aligned} \tag{36}
$$

Therefore, recalling (17), one obtains

$$
J_1=-\sum_{k=0}^{T-1}\left(\delta(k)-\tilde{\delta}(k)\right)^T \Delta_1(k)\left(\delta(k)-\tilde{\delta}(k)\right)+\left(e^{\xi}(0)\right)^T\left(\bar{R}(0)-\gamma_2^2(I_N\otimes W_1)\right)e^{\xi}(0). \tag{37}
$$

Noticing condition (31), apparently, the augmented state estimation error system (20) satisfies the H_∞ performance in Definition 2.

Subsequently, to develop an approach to solve the observer parameter sequence $\{Q(k)\}_{0\le k\le T-1}$, we consider the worst situation, that is $\delta(k)=\tilde{\delta}(k)=\tilde{G}(k)e^{\xi}(k)$, where $\tilde{G}(k)\triangleq\Delta_1^{-1}(k)Y^T(k)\bar{R}(k+1)L(k)$. Then, (20) can be converted into

$$
e^{\xi}(k+1)=\left(I_N\otimes M(k)+Y(k)\tilde{G}(k)\right)e^{\xi}(k)+\rho^{\xi}(k), \tag{38}
$$

where $\rho^{\xi}(k)=\left(I_N\otimes Q(k)\bar{C}(k)\right)e^{\xi}(k)$. Meanwhile, the following cost functional is constructed to represent the estimation effect of the observer:

$$
J_3\triangleq\mathbb{E}\left\{\sum_{k=0}^{T-1}\left(\left\|\bar{e}^{\xi}(k)\right\|^2+\gamma_3\left\|\rho^{\xi}(k)\right\|^2\right)\right\}, \tag{39}
$$

where γ_3 is typically a positive scalar.

Theorem 2: Consider singular system (3) with observer (5), and let the disturbance attenuation level γ_2, scalar $\gamma_3>0$ and the positive definite matrix W_2 be given. The state estimation error system (20) satisfies the H_∞ performance requirement (14) if there exist solutions $\left\{Q(k),\bar{R}(k),\bar{P}(k)\right\}_{0\leqslant k\leqslant T-1}$ (with condition $\bar{P}(T)=0$) satisfying the recursive RDEs (30) and the following one:

$$
\begin{aligned}
\bar{P}(k)=&\,V^T(k)V(k)+\left(I_N\otimes M(k)+Y(k)\tilde{G}(k)\right)^T\bar{P}(k+1)\left(I_N\otimes M(k)+Y(k)\tilde{G}(k)\right)\\
&+2\tilde{G}^T(k)Y^T(k)\bar{P}(k+1)\left(I_N\otimes Q(k)\bar{C}(k)\right)-(I_N\otimes M(k))^T\bar{P}(k+1)\Delta_2^{-1}(k)\bar{P}(k+1)(I_N\otimes M(k)),
\end{aligned} \tag{40}
$$

where $\Delta_2(k)\triangleq\gamma_3 I_{N(n_x+n_f)}+\bar{P}(k+1)$.

Moreover, the observer parameters $\{Q(k)\}_{0\leqslant k\leqslant T-1}$ can be calculated by

$$
Q(k)=\arg\min_{Q(k)}\left\|I_N\otimes Q(k)\bar{C}(k)+\Delta_2^{-1}(k)\bar{P}(k+1)(I_N\otimes M(k))\right\|_F. \tag{41}
$$

Proof: Based on Theorem 1, define the function as $\bar{\Psi}_2(k)\triangleq\mathbb{E}\left\{\left(e^{\xi}(k+1)\right)^T\bar{P}(k+1)e^{\xi}(k+1)-\left(e^{\xi}(k)\right)^T\times\bar{P}(k)e^{\xi}(k)\right\}$. Combining with (38), it is easy to deduce that

$$
\begin{aligned}
\bar{\Psi}_2(k)=&\left(e^{\xi}(k)\right)^T\left(\left(I_N\otimes M(k)+Y(k)\tilde{G}(k)\right)^T\bar{P}(k+1)\left(I_N\otimes M(k)+Y(k)\tilde{G}(k)\right)-\bar{P}(k)\right)e^{\xi}(k)\\
&+2\left(e^{\xi}(k)\right)^T\left(I_N\otimes M(k)+Y(k)\tilde{G}(k)\right)^T\bar{P}(k+1)\rho^{\xi}(k)+\left(\rho^{\xi}(k)\right)^T\bar{P}(k+1)\rho^{\xi}(k).
\end{aligned} \tag{42}
$$

Then, add zero term $\left\|\bar{e}^{\xi}(k)\right\|^2+\gamma_3\left\|\rho^{\xi}(k)\right\|^2-\left(\left\|\bar{e}^{\xi}(k)\right\|^2+\gamma_3\left\|\rho^{\xi}(k)\right\|^2\right)$ to the right side of (42), and it can be further written into

$$
\begin{aligned}
\bar{\Psi}_2(k)=&\left(e^{\xi}(k)\right)^T\left(\left(I_N\otimes M(k)+Y(k)\tilde{G}(k)\right)^T\bar{P}(k+1)\left(I_N\otimes M(k)+Y(k)\tilde{G}(k)\right)-\bar{P}(k)+V^T(k)V(k)\right)\\
&\times e^{\xi}(k)+2\left(e^{\xi}(k)\right)^T\left(I_N\otimes M(k)+Y(k)\tilde{G}(k)\right)^T\bar{P}(k+1)\rho^{\xi}(k)+\left(\rho^{\xi}(k)\right)^T\left(\gamma_3 I+\bar{P}(k+1)\right)\rho^{\xi}(k)\\
&-\left(\left\|\bar{e}^{\xi}(k)\right\|^2+\gamma_3\left\|\rho^{\xi}(k)\right\|^2\right)\\
=&\left(e^{\xi}(k)\right)^T\left(\left(I_N\otimes M(k)+Y(k)\tilde{G}(k)\right)^T\bar{P}(k+1)\left(I_N\otimes M(k)+Y(k)\tilde{G}(k)\right)-\bar{P}(k)+V^T(k)V(k)\right.\\
&\left.+2\left(Y(k)\tilde{G}(k)\right)^T\bar{P}(k+1)\left(I_N\otimes Q(k)\bar{C}(k)\right)\right)e^{\xi}(k)+2\left(e^{\xi}(k)\right)^T(I_N\otimes M(k))^T\bar{P}(k+1)\rho^{\xi}(k)\\
&+\left(\rho^{\xi}(k)\right)^T\left(\gamma_3 I+\bar{P}(k+1)\right)\rho^{\xi}(k)-\left(\left\|\bar{e}^{\xi}(k)\right\|^2+\gamma_3\left\|\rho^{\xi}(k)\right\|^2\right).
\end{aligned} \tag{43}
$$

It readily follows that, based on completing squares method, (43) is equivalent to

$$\begin{aligned}\bar{\Psi}_2(k) = &\left(e^{\xi}(k)\right)^T \left(\left(I_N \otimes M(k) + Y(k)\tilde{G}(k)\right)^T \bar{P}(k+1)\left(I_N \otimes M(k) + Y(k)\tilde{G}(k)\right) - \bar{P}(k) + V^T(k)V(k)\right.\\ &\left.+2\left(Y(k)\tilde{G}(k)\right)^T \bar{P}(k+1)\left(I_N \otimes Q(k)\bar{C}(k)\right)\right) e^{\xi}(k) + \left(\rho^{\xi}(k) + \tilde{\rho}^{\xi}(k)\right)^T \Delta_2(k)\left(\rho^{\xi}(k) + \tilde{\rho}^{\xi}(k)\right)\\ &- \left(\tilde{\rho}^{\xi}(k)\right)^T \Delta_2(k)\tilde{\rho}^{\xi}(k) - \left(\left\|\bar{e}^{\xi}(k)\right\|^2 + \gamma_3\left\|\rho^{\xi}(k)\right\|^2\right),\end{aligned} \tag{44}$$

where $\tilde{\rho}^{\xi}(k) \triangleq \Delta_2^{-1}(k)\bar{P}(k+1)(I_N \otimes M(k))e^{\xi}(k)$.

Substituting (40) into (44) and summing up it from 0 to T, it follows

$$\begin{aligned}&\left(e^{\xi}(T)\right)^T \bar{P}(T)e^{\xi}(T) - \left(e^{\xi}(0)\right)^T \bar{P}(0)e^{\xi}(0)\\ &= \sum_{k=0}^{T-1}\left(\rho^{\xi}(k) + \tilde{\rho}^{\xi}(k)\right)^T \Delta_2(k)\left(\rho^{\xi}(k) + \tilde{\rho}^{\xi}(k)\right) - \sum_{k=0}^{T-1}\left(\left\|\bar{e}^{\xi}(k)\right\|^2 + \gamma_3\left\|\rho^{\xi}(k)\right\|^2\right).\end{aligned} \tag{45}$$

Noticing the cost function (39), it follows that

$$J_3(k) = \sum_{k=0}^{T-1}\left(\rho^{\xi}(k) + \tilde{\rho}^{\xi}(k)\right)^T \Delta_2(k)\left(\rho^{\xi}(k) + \tilde{\rho}^{\xi}(k)\right) + \left(e^{\xi}(0)\right)^T \bar{P}(0)e^{\xi}(0). \tag{46}$$

Recalling the definitions of $\rho^{\xi}(k)$ and $\tilde{\rho}^{\xi}(k)$, one obtains

$$\begin{aligned}J_3(k) \leqslant &\sum_{k=0}^{T-1}\left\|(I_N \otimes Q(k))\left(I_N \otimes \bar{C}(k)\right) + \Delta_2^{-1}(k)\bar{P}(k+1)(I_N \otimes M(k))\right\|_F^2 \|\Delta_2(k)\|_F\left\|e^{\xi}(k)\right\|^2\\ &+ \left(e^{\xi}(0)\right)^T \bar{P}(0)e^{\xi}(0).\end{aligned} \tag{47}$$

Apparently, the observer gain $\{Q(k)\}_{0\le k\le T-1}$ should make the cost function J_3 as small as possible. A method of calculating $Q(k)$ is given below.

Denoting $I_N \otimes \bar{C}(k) \triangleq \left[\kappa_1^T(k), \ldots, \kappa_N^T(k)\right]^T$, $\Delta_2^{-1}(k)\bar{P}(k+1)(I_N \otimes M(k)) \triangleq \left[\psi_1^T(k), \ldots, \psi_N^T(k)\right]^T$ and considering (41), it can be found that

$$\begin{aligned}Q(k) &= \arg\min_{Q(k)}\left\|I_N \otimes Q(k)\bar{C}(k) + \Delta_2^{-1}(k)\bar{P}(k+1)(I_N \otimes M(k))\right\|_F\\ &= \arg\min_{Q(k)}\left\|\begin{bmatrix} Q(k)\kappa_1(k) + \psi_1(k)\\ Q(k)\kappa_2(k) + \psi_2(k)\\ \vdots\\ Q(k)\kappa_N(k) + \psi_N(k)\end{bmatrix}\right\|_F\\ &= \arg\min_{Q(k)}\left\|Q(k)\left[\kappa_1(k),\ \kappa_2(k), \cdots, \kappa_N(k)\right] + \left[\psi_1(k),\ \psi_2(k), \cdots, \psi_N(k)\right]\right\|_F.\end{aligned}$$

In light of Lemma 4, $Q(k)$ can be obtained from the following equation:

$$Q(k) = -\left[\psi_1(k),\ \psi_2(k), \cdots, \psi_N(k)\right] \times \left[\kappa_1(k),\ \kappa_2(k), \cdots, \kappa_N(k)\right]^{\dagger}, \tag{48}$$

which is the solution of the optimization problem (41). Recalling Theorem 1, (20) satisfies the H_∞ performance (14) under the influence of designed $Q(k)$, and the proof is complete.

So far, we have designed the observer (5) for the augmented singular system (3), and given the conditions to ensure that the augmented estimation error achieves the H_∞ performance given in Definition 2, thus the simultaneous estimation of state and fault vectors is realized. Considering the SCP, based on the above work, the following will deal with the problem of designing controller parameters for the fault-tolerant formation consensus controller as shown in (48).

3.2. Controller Parameters Design

Theorem 3: Consider MAS (3) with the observer (5) and the fault-tolerant controller (10), and let the disturbance attenuation level $\{\gamma_1, \gamma_2\}$, the scalar γ_4 and the positive definite matrix $\{W_1, W_2\}$ be given. The time-varying system (28) satisfies the H_∞ performance requirement (29) if there exist solutions $\{Q(k), \bar{R}(k), K(k), \bar{K}(k), \bar{S}(k)\}_{0\leqslant k\leqslant T-1}$ (with condition $\bar{S}(T) = 0$) satisfying (30) and the following recursive RDE:

$$\begin{aligned}\bar{S}(k) = &\hat{V}^T(k)\hat{V}(k) + \tilde{A}^T(k)\bar{S}(k+1)\tilde{A}(k) + 2\Omega_1(k+1)\tilde{A}(k) + \Omega_2(k+1)\\ &+ \left(\tilde{A}^T(k)\bar{S}(k+1)\bar{H}(k) + \Omega_1(k+1)\bar{H}(k) + \tilde{A}^T(k)\Omega_4^T(k+1) + \Omega_3(k+1)\right)\\ &\times \Delta_3^{-1}(k)\left(\tilde{A}^T(k)\bar{S}(k+1)\bar{H}(k) + \Omega_1(k+1)\bar{H}(k) + \tilde{A}^T(k)\Omega_4^T(k+1) + \Omega_3(k+1)\right)^T\\ &- \left(\tilde{A}^T(k)\bar{S}(k+1)\bar{N}(k) + \Omega_1(k+1)\bar{N}(k)\right)\Delta_4^{-1}(k)\left(\tilde{A}^T(k)\bar{S}(k+1)\bar{N}(k) + \Omega_1(k+1)\bar{N}(k)\right)^T,\end{aligned} \tag{49}$$

subject to

$$\Delta_3(k) \triangleq \gamma_1^2 I_{N(n_x+n_f+n_w)} - \bar{H}^T(k)\bar{S}(k+1)\bar{H}(k) - 2\Omega_4(k+1)\bar{H}(k) - \Omega_5(k+1) > 0, \tag{50}$$

$$\Delta_4(k) \triangleq \left(\gamma_1^2 - \gamma_4^2\right) I_{Nn_x} > 0, \tag{51}$$

$$\Delta_5(k) \triangleq \gamma_4^2 I_{Nn_x} - \bar{N}^T(k)\bar{S}(k+1)\bar{N}(k) > 0, \tag{52}$$

$$\bar{S}(0) < \gamma_1^2 (I_N \otimes W_1), \tag{53}$$

$$\Upsilon(k) \triangleq 2\rho(F(k)) \left(\left\|e^{\xi}(k)\right\|^2 + \varpi^2\right)^{\frac{1}{2}} - \|\tilde{w}(k)\| \leqslant 0, \tag{54}$$

where

$$\begin{aligned}
\Omega_1(k+1) &\triangleq \mathbb{E}\left\{\tilde{B}_{h(k)}^T(k)\bar{S}(k+1)\right\} = \sum_{i\in R} \bar{p}_i(k)\tilde{B}_i^T(k)\bar{S}(k+1),\\
\Omega_2(k+1) &\triangleq \mathbb{E}\left\{\tilde{B}_{h(k)}^T(k)\bar{S}(k+1)\tilde{B}_{h(k)}(k)\right\} = \sum_{i\in R} \bar{p}_i(k)\tilde{B}_i^T(k)\bar{S}(k+1)\tilde{B}_i(k),\\
\Omega_3(k+1) &\triangleq \mathbb{E}\left\{\tilde{B}_{h(k)}^T(k)\bar{S}(k+1)\bar{M}_{h(k)}(k)\right\} = \sum_{i\in R} \bar{p}_i(k)\tilde{B}_i^T(k)\bar{S}(k+1)\bar{M}_i(k),\\
\Omega_4(k+1) &\triangleq \mathbb{E}\left\{\bar{M}_{h(k)}^T(k)\bar{S}(k+1)\right\} = \sum_{i\in R} \bar{p}_i(k)\bar{M}_i^T(k)\bar{S}(k+1),\\
\Omega_5(k+1) &\triangleq \mathbb{E}\left\{\bar{M}_{h(k)}^T(k)\bar{S}(k+1)\bar{M}_{h(k)}(k)\right\} = \sum_{i\in R} \bar{p}_i(k)\bar{M}_i^T(k)\bar{S}(k+1)\bar{M}_i(k),\\
F(k) &\triangleq \Delta_5^{-1}(k)\left(\gamma_3^2 I + \bar{N}^T(k)\bar{S}(k+1)\bar{H}(k) + \bar{N}^T(k)\Omega_4^T(k+1)\right).
\end{aligned}$$

Proof: Define $\mho_e(k) = (e^c(k))^T\bar{S}(k)e^c(k)$ and calculate the expectation of $\mho_e(k+1) - \mho_e(k)$ as follows:

$$\begin{aligned}
\bar{\Psi}_3(k) &\triangleq \mathbb{E}\{\mho_e(k+1) - \mho_e(k)\}\\
&= \mathbb{E}\left\{\left(\left(\tilde{A}(k) + \tilde{B}_{h(k)}(k)\right)e^c(k) + \left(\bar{H}(k) + \bar{M}_{h(k)}(k)\right)\bar{\eta}(k) + \bar{N}(k)\tilde{w}(k)\right)^T\bar{S}(k+1)\right.\\
&\quad \times \left.\left(\left(\tilde{A}(k) + \tilde{B}_{h(k)}(k)\right)e^c(k) + \left(\bar{H}(k) + \bar{M}_{h(k)}(k)\right)\bar{\eta}(k) + \bar{N}(k)\tilde{w}(k)\right) - (e^c(k))^T\bar{S}(k)e^c(k)\right\}\\
&= \mathbb{E}\left\{(e^c(k))^T\left(\tilde{A}(k) + \tilde{B}_{h(k)}(k)\right)^T\bar{S}(k+1)\left(\tilde{A}(k) + \tilde{B}_{h(k)}(k)\right)e^c(k) + 2(e^c(k))^T\left(\tilde{A}(k) + \tilde{B}_{h(k)}(k)\right)^T\right.\\
&\quad \times \bar{S}(k+1)\left(\bar{H}(k) + \bar{M}_{h(k)}(k)\right)\bar{\eta}(k) + \bar{\eta}^T(k)\left(\bar{H}(k) + \bar{M}_{h(k)}(k)\right)^T\bar{S}(k+1)\left(\bar{H}(k) + \bar{M}_{h(k)}(k)\right)\bar{\eta}(k)\\
&\quad + 2(e^c(k))^T\left(\tilde{A}(k) + \tilde{B}_{h(k)}(k)\right)^T\bar{S}(k+1)\bar{N}(k)\tilde{w}(k) + 2\bar{\eta}^T(k)\left(\bar{H}(k) + \bar{M}_{h(k)}(k)\right)^T\bar{S}(k+1)\bar{N}(k)\tilde{w}(k)\\
&\quad \left. + \tilde{w}^T(k)\bar{N}^T(k)\bar{S}(k+1)\bar{N}(k)\tilde{w}(k) - (e^c(k))^T\bar{S}(k)e^c(k)\right\}.
\end{aligned} \tag{55}$$

Then (55) can be further written into

$$\begin{aligned}
\bar{\Psi}_3(k) = \mathbb{E}&\left\{(e^c(k))^T\left(\tilde{A}^T(k)\bar{S}(k+1)\tilde{A}(k) + 2\Omega_1(k+1)\tilde{A}(k) + \Omega_2(k+1) - \bar{S}(k)\right)e^c(k)\right.\\
&+ 2(e^c(k))^T\left(\tilde{A}^T(k)\bar{S}(k+1)\bar{H}(k) + \Omega_1(k+1)\bar{H}(k) + \tilde{A}^T(k)\Omega_4^T(k+1) + \Omega_3(k+1)\right)\bar{\eta}(k)\\
&+ \bar{\eta}^T(k)\left(\bar{H}^T(k)\bar{S}(k+1)\bar{H}(k) + 2\Omega_4(k+1)\bar{H}(k) + \Omega_5(k+1)\right)\bar{\eta}(k)\\
&+ 2(e^c(k))^T\left(\tilde{A}^T(k)\bar{S}(k+1)\bar{N}(k) + \Omega_1(k+1)\bar{N}(k)\right)\tilde{w}(k)\\
&\left. + 2\bar{\eta}^T(k)\left(\bar{H}^T(k)\bar{S}(k+1)\bar{N}(k) + \Omega_4(k+1)\bar{N}(k)\right)\tilde{w}(k) + \tilde{w}^T(k)\bar{N}^T(k)\bar{S}(k+1)\bar{N}(k)\tilde{w}(k)\right\}.
\end{aligned} \tag{56}$$

Adding the zero term $\|\bar{e}^c(k)\|^2 - \gamma_1^2\|\bar{\eta}(k) - \tilde{w}(k)\|^2 - \left(\|\bar{e}^c(k)\|^2 - \gamma_1^2\|\bar{\eta}(k) - \tilde{w}(k)\|^2\right)$ to (56), it follows

$$\begin{aligned}
\bar{\Psi}_3(k) = \mathbb{E}&\left\{(e^c(k))^T\left(\tilde{A}^T(k)\bar{S}(k+1)\tilde{A}(k) + 2\Omega_1(k+1)\tilde{A}(k) + \Omega_2(k+1) - \bar{S}(k) + \hat{V}^T(k)\hat{V}(k)\right)e^c(k)\right.\\
&+ 2(e^c(k))^T\left(\tilde{A}^T(k)\bar{S}(k+1)\bar{H}(k) + \Omega_1(k+1)\bar{H}(k) + \tilde{A}^T(k)\Omega_4^T(k+1) + \Omega_3(k+1)\right)\bar{\eta}(k)\\
&- \bar{\eta}^T(k)\left(\gamma_1^2 I - \bar{H}^T(k)\bar{S}(k+1)\bar{H}(k) - 2\Omega_4(k+1)\bar{H}(k) - \Omega_5(k+1)\right)\bar{\eta}(k)\\
&+ 2(e^c(k))^T\left(\tilde{A}^T(k)\bar{S}(k+1)\bar{N}(k) + \Omega_1(k+1)\bar{N}(k)\right)\tilde{w}(k) - \left(\gamma_1^2 - \gamma_4^2\right)\tilde{w}^T(k)\tilde{w}(k)\\
&+ 2\bar{\eta}^T(k)\left(\bar{H}^T(k)\bar{S}(k+1)\bar{N}(k) + \Omega_4(k+1)\bar{N}(k)\right)\tilde{w}(k) - \tilde{w}^T(k)\left(\gamma_4^2 I - \bar{N}^T(k)\bar{S}(k+1)\bar{N}(k)\right)\tilde{w}(k)\\
&\left. - \left(\|\bar{e}^c(k)\|^2 - \gamma_1^2\|\bar{\eta}(k) - \tilde{w}(k)\|^2\right)\right\}.
\end{aligned} \tag{57}$$

Applying the completing squares method, one obtains

$$\begin{aligned}\bar{\Psi}_3(k)=\mathbb{E}\Big\{&(e^c(k))^T\left(\tilde{A}^T(k)\bar{S}(k+1)\tilde{A}(k)+2\Omega_1(k+1)\tilde{A}(k)+\Omega_2(k+1)-\bar{S}(k)+\hat{V}^T(k)\hat{V}(k)\right)e^c(k)\\&+\tilde{\eta}^T(k)\Delta_3(k)\tilde{\eta}(k)-(\bar{\eta}(k)-\tilde{\eta}(k))^T\Delta_3(k)(\bar{\eta}(k)-\tilde{\eta}(k))+\hat{w}^T(k)\Delta_4(k)\hat{w}(k)\\&-(\tilde{w}(k)-\hat{w}(k))^T\Delta_4(k)(\tilde{w}(k)-\hat{w}(k))+\check{w}^T(k)\Delta_5(k)\check{w}(k)-(\tilde{w}(k)-\check{w}(k))^T\Delta_5(k)(\tilde{w}(k)-\check{w}(k))\\&-\left(\|\bar{e}^c(k)\|^2-\gamma_1^2\|\bar{\eta}(k)-\tilde{w}(k)\|^2\right)\Big\},\end{aligned}\tag{58}$$

where

$$\begin{aligned}&\tilde{\eta}(k)\triangleq\tilde{F}(k)e^c(k),\tilde{F}(k)\triangleq\Delta_3^{-1}(k)\left(\bar{H}^T(k)\bar{S}(k+1)\tilde{A}(k)+\Omega_4(k+1)\tilde{A}(k)+\bar{H}^T(k)\Omega_1^T(k+1)+\Omega_3^T(k+1)\right),\\&\hat{w}(k)\triangleq\hat{F}(k)e^c(k),\hat{F}(k)\triangleq\Delta_4^{-1}(k)\left(\bar{N}^T(k)\bar{S}(k+1)\tilde{A}(k)+\bar{N}^T(k)\Omega_1^T(k+1)\right),\check{w}(k)\triangleq F(k)\bar{\eta}(k).\end{aligned}$$

Noticing (49), it follows from (29) that

$$\begin{aligned}J_2=\mathbb{E}\Bigg\{&-\sum_{k=0}^{T-1}\Big((\bar{\eta}(k)-\tilde{\eta}(k))^T\Delta_3(k)(\bar{\eta}(k)-\tilde{\eta}(k))-(\tilde{w}(k)-\hat{w}(k))^T\Delta_4(k)(\tilde{w}(k)-\hat{w}(k))\\&-(\tilde{w}(k)-\check{w}(k))^T\Delta_5(k)(\tilde{w}(k)-\check{w}(k))+\check{w}^T(k)\Delta_5(k)\check{w}(k)\Big)+(e^c(0))^T\left(\bar{S}(0)-\gamma_1^2(I_N\otimes W_1)\right)e^c(0)\Bigg\}.\end{aligned}\tag{59}$$

Noticing the definition of $\bar{\eta}(k)=\left[\left(e^{\xi}(k)\right)^T\quad w^T(k)\right]^T$, it can be inferred that

$$\Upsilon(k)\geqslant 2\rho(F(k))\|\bar{\eta}(k)\|-\|\tilde{w}(k)\|\geqslant 2\|F(k)\|\cdot\|\bar{\eta}(k)\|-\|\tilde{w}(k)\|\geqslant 2\|\check{w}(k)\|-\|\tilde{w}(k)\|.\tag{60}$$

Owing to the condition (54), we have $2\|\check{w}(k)\|\cdot\|\tilde{w}(k)\|-\|\tilde{w}(k)\|^2\leqslant 0$, which is equivalent to $\|\check{w}(k)\|^2-(\|\tilde{w}(k)\|-\|\check{w}(k)\|)^2\leqslant 0$. Since $(\|\tilde{w}(k)\|-\|\check{w}(k)\|)^2\leqslant\|\tilde{w}(k)-\check{w}(k)\|^2$, it readily follows that $\|\check{w}(k)\|^2-\|\tilde{w}(k)-\check{w}(k)\|^2\leqslant 0$, which indicates that $\check{w}^T(k)\Delta_5(k)\check{w}(k)-(\tilde{w}(k)-\check{w}(k))^T\Delta_5(k)(\tilde{w}(k)-\check{w}(k))\leqslant 0$. Therefore, under the condition of (53), the H_∞ performance constraint defined by (29) is satisfied over the finite horizon $[0,T-1]$. A sufficient condition is established for time-varying MAS (2) achieving formation consensus, and the proof is complete.

Remark 3: Both J_1 and J_2 are H_∞ performance constraints over the finite horizon ($k\in[0,T-1]$), the difference being that they are designed for different systems. Specifically, if $J_1<0$, then the augmented state estimation error system (20) satisfies the H_∞ performance constraint. If $J_2<0$, the formation consensus error system (28) satisfies the H_∞ performance constraint, which is equivalent to MAS achieving fault-tolerant H_∞ formation consensus control. Therefore, J_1 is used to prove Theorem 1 for solving the time-varying parameter matrices $\{Q(k)\}_{0\leqslant k\leqslant T-1}$ of the state observer (16). And J_2 is used to prove Theorem 3 for solving $\left\{K(k),\bar{K}(k)\right\}_{0\leqslant k\leqslant T-1}$ of the fault-tolerant controller (11).

In Theorem 3, we discuss the condition that the formation consensus error system (28) satisfies the H_∞ performance by resorting to backward recursive RDEs. In order to give a solution to the controller gain $\{K(k)\}_{0\leqslant k\leqslant T-1}$, rewrite (28) as follows:

$$e^c(k+1)=\left(\tilde{A}(k)+\tilde{B}_{h(k)}(k)\right)e^c(k)+\left(\tilde{H}(k)+\tilde{M}_{h(k)}(k)\right)\eta(k),\tag{61}$$

where $\tilde{H}(k)\triangleq\left[\bar{H}(k)\quad\bar{N}(k)\right]$, $\tilde{M}_{h(k)}(k)\triangleq\left[\bar{M}_{h(k)}(k)\quad 0_{Nn_x\times Nn_x}\right]$, $\eta(k)\triangleq\left[\bar{\eta}^T(k)\quad\tilde{w}^T(k)\right]^T$. Consider the particular case, that is, specifying $\eta(k)$ has a special form $\eta(k)=\tilde{V}(k)e^c(k)$, where $\tilde{V}(k)\triangleq\left[\tilde{F}^T(k)\quad\hat{F}^T(k)\right]^T$.

Subsequently, from (61), the formation consensus error system in this case can be obtained:

$$e^c(k+1)=\left(\hat{A}(k)+\tilde{H}(k)\tilde{V}(k)\right)e^c(k)+\left(\bar{B}_{h(k)}(k)+\tilde{M}_{h(k)}(k)\tilde{V}(k)\right)e^c(k)+\hat{B}(k)\rho^c(k),\tag{62}$$

where $\hat{A}(k)\triangleq I_N\otimes A(k)$, $\hat{B}(k)\triangleq K\bar{\Lambda}(k)\otimes B(k)$, $\rho^c(k)=\tilde{K}(k)e^c(k)$, $\tilde{K}(k)\triangleq I_N\otimes K(k)$.

Define the following cost functional:

$$J_4\triangleq\mathbb{E}\left\{\sum_{k=0}^{T-1}\left(\|\bar{e}^c(k)\|^2+\gamma_5\|\rho^c(k)\|^2\right)\right\},\tag{63}$$

where $\sum_{k=0}^{T-1}\|\bar{e}^c(k)\|^2$ represents the control performance, $\sum_{k=0}^{T-1}\|\rho^c(k)\|^2$ stands for the control effort, and scalar $\gamma_5>0$ is selected according to special needs.

Theorem 4: Consider MAS (3) with the observer (5) and controller (10), and let the disturbance attenuation level $\{\gamma_1,\gamma_2\}$, the scalar $\gamma_5>0$ and the positive definite matrix $\{W_1,W_2\}$ be given. The MAS achieves formation consensus if there exist solutions $\left\{Q(k),\bar{R}(k),K(k),\bar{K}(k),\bar{S}(k),\bar{Y}(k)\right\}_{0\leqslant k\leqslant T-1}$ [with condition $\bar{Y}(T)=0$] satisfying (30), (49) and the following recursive RDE:

$$\begin{aligned}\bar{Y}(k) &= \hat{V}^T(k)\hat{V}(k) + \left(\hat{A}(k)+\tilde{H}(k)\tilde{V}(k)\right)^T \bar{Y}(k+1)\left(\hat{A}(k)+\tilde{H}(k)\tilde{V}(k)\right) + \Omega_7(k+1)\\ &\quad + 2\Omega_6(k+1)\left(\hat{A}(k)+\tilde{H}(k)\tilde{V}(k)\right) + 2\tilde{V}^T(k)\tilde{H}^T(k)\bar{Y}(k+1)\hat{B}(k)\tilde{K}(k)\\ &\quad + 2\Omega_6(k+1)\hat{B}(k)\tilde{K}(k) - \hat{A}^T(k)\bar{Y}(k+1)\hat{B}(k)\Delta_6^{-1}(k)\hat{B}^T(k)\bar{Y}(k+1)\hat{A}(k),\end{aligned} \tag{64}$$

subject to

$$\Delta_6(k) \triangleq \gamma_5 I_{Nn_u} + \hat{B}^T(k)\bar{Y}(k+1)\hat{B}(k) > 0, \tag{65}$$

where

$$\begin{aligned}\Omega_6(k+1) &\triangleq \mathbb{E}\left\{\left(\tilde{B}_{h(k)}(k)+\tilde{M}_{h(k)}(k)\tilde{V}(k)\right)^T\bar{Y}(k+1)\right\} = \sum_{i\in R}\bar{p}_i(k)\left(\tilde{B}_i(k)+\tilde{M}_i(k)\tilde{V}(k)\right)^T\bar{Y}(k+1),\\ \Omega_7(k+1) &\triangleq \mathbb{E}\left\{\left(\tilde{B}_{h(k)}(k)+\tilde{M}_{h(k)}(k)\tilde{V}(k)\right)^T\bar{Y}(k+1)\left(\tilde{B}_{h(k)}(k)+\tilde{M}_{h(k)}(k)\tilde{V}(k)\right)\right\}\\ &= \sum_{i\in R}\bar{p}_i(k)\left(\tilde{B}_i(k)+\tilde{M}_i(k)\tilde{V}(k)\right)^T\bar{Y}(k+1)\left(\tilde{B}_i(k)+\tilde{M}_i(k)\tilde{V}(k)\right).\end{aligned}$$

Besides, the controller gain can be obtained by the following equation:

$$K(k) = \arg\min_{K(k)}\left\|\tilde{K}(k)+\Delta_6^{-1}(k)\hat{B}^T(k)\bar{Y}(k+1)\hat{A}(k)\right\|_F. \tag{66}$$

Proof: Similar to the proof process of Theorem 2, we define $\bar{\Psi}_4(k) \triangleq \mathbb{E}\left\{(e^c(k+1))^T\bar{Y}(k+1)e^c(k+1) -(e^c(k))^T\bar{Y}(k)e^c(k)\right\}$ based on Theorem 3, and recalling (62) yields

$$\begin{aligned}\bar{\Psi}_4(k) &= \mathbb{E}\Big\{\Big((e^c(k))^T\left(\hat{A}(k)+\tilde{H}(k)\tilde{V}(k)\right)^T + (e^c(k))^T\left(\tilde{B}_{h(k)}(k)+\tilde{M}_{h(k)}(k)\tilde{V}(k)\right)^T + (\rho^c(k))^T\hat{B}^T(k)\Big)\\ &\quad\times\bar{Y}(k+1)\Big(\left(\hat{A}(k)+\tilde{H}(k)\tilde{V}(k)\right)e^c(k) + \left(\tilde{B}_{h(k)}(k)+\tilde{M}_{h(k)}(k)\tilde{V}(k)\right)e^c(k) + \hat{B}(k)\rho^c(k)\Big)\\ &\quad-(e^c(k))^T\bar{Y}(k)e^c(k)\Big\}\\ &= \mathbb{E}\Big\{(e^c(k))^T\Big(\left(\hat{A}(k)+\tilde{H}(k)\tilde{V}(k)\right)^T\bar{Y}(k+1)\left(\hat{A}(k)+\tilde{H}(k)\tilde{V}(k)\right) + 2\left(\tilde{B}_{h(k)}(k)+\tilde{M}_{h(k)}(k)\tilde{V}(k)\right)^T\\ &\quad\times\bar{Y}(k+1)\left(\hat{A}(k)+\tilde{H}(k)\tilde{V}(k)\right) + \left(\tilde{B}_{h(k)}(k)+\tilde{M}_{h(k)}(k)\tilde{V}(k)\right)^T\bar{Y}(k+1)\left(\tilde{B}_{h(k)}(k)+\tilde{M}_{h(k)}(k)\tilde{V}(k)\right)\\ &\quad-\bar{Y}(k)\Big)e^c(k) + 2(e^c(k))^T\Big(\left(\hat{A}(k)+\tilde{H}(k)\tilde{V}(k)\right)^T\bar{Y}(k+1)\hat{B}(k) + \left(\tilde{B}_{h(k)}(k)+\tilde{M}_{h(k)}(k)\tilde{V}(k)\right)^T\\ &\quad\times\bar{Y}(k+1)\hat{B}(k)\Big)\rho^c(k) + (\rho^c(k))^T\hat{B}^T(k)\bar{Y}(k+1)\hat{B}(k)\rho^c(k)\Big\}.\end{aligned} \tag{67}$$

Noticing the definitions of $\Omega_6(k+1)$ and $\Omega_7(k+1)$, add the zero term $\|\bar{e}^c(k)\|^2+\gamma_5\|\rho^c(k)\|^2-\left(\|\bar{e}^c(k)\|^2 -\gamma_5\|\rho^c(k)\|^2\right)$ to (67), then it can be equivalently expressed as

$$\begin{aligned}\bar{\Psi}_4(k) &= \mathbb{E}\Big\{(e^c(k))^T\Big(\left(\hat{A}(k)+\tilde{H}(k)\tilde{V}(k)\right)^T\bar{Y}(k+1)\left(\hat{A}(k)+\tilde{H}(k)\tilde{V}(k)\right) + 2\Omega_6(k+1)\left(\hat{A}(k)+\tilde{H}(k)\tilde{V}(k)\right)\\ &\quad+\Omega_7(k+1) - \bar{Y}(k) + 2\tilde{V}^T(k)\tilde{H}^T(k)\bar{Y}(k+1)\hat{B}(k)\tilde{K}(k) + 2\Omega_6(k+1)\hat{B}(k)\tilde{K}(k) + \hat{V}^T(k)\hat{V}(k)\Big)e^c(k)\\ &\quad+(\rho^c(k)+\tilde{\rho}^c(k))^T\Delta_6(k)(\rho^c(k)+\tilde{\rho}^c(k)) - (\tilde{\rho}^c(k))^T\Delta_6(k)\tilde{\rho}^c(k) - \left(\|\bar{e}^c(k)\|^2+\gamma_5\|\rho^c(k)\|^2\right)\Big\},\end{aligned} \tag{68}$$

where $\tilde{\rho}^c(k) \triangleq \Delta_6^{-1}(k)\hat{B}^T(k)\bar{Y}(k+1)\hat{A}(k)e^c(k)$.

Subsequently, it follows from (64) and the cost functional (63) that

$$\begin{aligned}J_4 &= \left\{\sum_{k=0}^{T-1}\left((\rho^c(k)+\tilde{\rho}^c(k))^T\Delta_6(k)(\rho^c(k)+\tilde{\rho}^c(k))\right) + (e^c(0))^T\bar{Y}(0)e^c(0)\right\}\\ &\leqslant \sum_{k=0}^{T-1}\left\{\left\|\tilde{K}(k)+\Delta_6^{-1}(k)\hat{B}^T(k)\bar{Y}(k+1)\hat{A}(k)\right\|_F^2\|\Delta_6(k)\|_F\|e^c(k)\|^2 + \left(e^{\xi}(0)\right)^T\bar{Y}(0)e^{\xi}(0)\right\}.\end{aligned} \tag{69}$$

Algorithm I: Fault-tolerant formation consensus controller algorithm.

Step 1. Let scalars $\{\gamma_1,\gamma_2,\gamma_4\}$, $\{\gamma_3,\gamma_5\}>0$ and positive definite matrix $\{W_1,W_2\}$ be given. Set $k=T-1$, and $\bar{R}(T)=\bar{P}(T)=\bar{S}(T)=\bar{Y}(T)=0$.

Step 2. Calculate $Q(k)$ by (48), then compute $\Delta_1(k)$ by (32). If $\Delta_1(k)>0$, then go to the next step, else jump to Step 7.

Step 3. Solve the backward RDEs (30) and (40) to get $\bar{R}(k)$ and $\bar{P}(k)$, respectively.

Step 4. Compute $\Delta_6(k)$ by (65). If $\Delta_6(k)>0$, then calculate $K(k)$ by (70), else jump to Step 7.

Step 5. Compute $\Delta_3(k)$ and $\Upsilon(k)$ respectively by (50) and (54). If $\Delta_3(k)>0$ and $\Upsilon(k)\le 0$, compute $\bar{K}(k)$ satisfying (52), else jump to Step 7.

Step 6. Solve the backward RDEs (49) and (64) to get $\bar{S}(k)$ and $\bar{Y}(k)$, respectively. If $k\neq 0$, then set $k=k-1$ and go back to Step 2, else go to the next step.

Step 7. If conditions $\{\Delta_1(k),\Delta_3(k),\Delta_6(k)\}>0,\Upsilon(k)\le 0,\bar{R}(0)<\gamma_2^2(I_N\otimes W_2)$ and $\bar{S}(0)<\gamma_1^2(I_N\otimes W_1)$ are not satisfied, this algorithm is infeasible. Stop.

According to the definition of cost functional in (63), the controller gain $K(k)$ obtained by (66) is the best choice to suppress J_4. Considering that $\tilde{K}(k)=I_N\otimes K(k)$ has a special structure, similar to the calculation method of observer gain $Q(k)$, we introduce the notation

$$I_{Nn_x}\triangleq\left[\mu_1^T(k),\ \mu_2^T(k),\cdots,\mu_N^T(k)\right]^T,\Delta_6^{-1}(k)\hat{B}^T(k)\bar{Y}(k+1)\hat{A}(k)\triangleq\left[\sigma_1^T(k),\ \sigma_2^T(k),\cdots,\sigma_N^T(k)\right]^T.$$

It readily follows from Lemma 4 that $K(k)$ can be obtained by

$$K(k)=-[\sigma_1(k),\ \sigma_2(k),\cdots,\sigma_N(k)]\times\left[\mu_1(k),\ \mu_2(k),\cdots,\mu_N(k)\right]^{\dagger}. \tag{70}$$

Recalling Theorem 3, it can be seen that under the effect of fault-tolerant controller (11), the MAS can achieve the formation consensus defined in Definition 3.

Based on the above analysis, the finite-horizon fault-tolerant formation consensus control design algorithm can be summarized.

4. Illustrative Example

In this section, a numerical example is provided to validate the effectiveness of the developed scheme of distributed H_∞ fault-tolerant formation consensus controller design. Consider a time-varying MAS that consists of five agents, and the dynamics of agents are modelled as (2) with following parameters:

$$A(k)=\begin{bmatrix}0.99+0.05\cos(0.4k) & -0.45\\ -0.10 & -0.73-0.1\cos(0.5k)\end{bmatrix},B(k)=\begin{bmatrix}0.2-0.05\sin(0.3k)\\ 0.2\end{bmatrix},E(k)=\begin{bmatrix}0.06 & 0.24\end{bmatrix}^T,$$
$$E_f(k)=\begin{bmatrix}1 & 1\end{bmatrix}^T,C(k)=\begin{bmatrix}0.105 & 0.082\end{bmatrix},D(k)=-0.3,F_f(k)=1,\bar{V}(k)=0.0015I.$$

In the simulation, it is assumed that the topology of MAS is denoted by a directed graph G shown in Figure 2 with all the edge weight as 1. Assume that the following faults occur simultaneously on agents 1 and 3:

$$f_1(k)=\begin{cases}0, & k<85\\ -0.3\sin(0.1k+10), & \text{otherwise},\end{cases}$$
$$f_3(k)=\begin{cases}0, & k<85\\ 0.4, & \text{otherwise},\end{cases}$$
$$f_l(k)=0,\ l=2,4,5.$$

The disturbance input $\{w_l(k),v_l(k)\}(l=1,2,\cdots,5)$ are respectively selected as $w_l(k)=0.05(\sin(b_l^wk)+\cos(c_l^wk))$ and $v_l(k)=0.05\left(\sin(b_l^vk)+\cos(c_l^vk)\right)$, where $\left\{b_l^w,c_l^w,b_l^v,c_l^v\right\}(l=1,2,\cdots,5)$ are sequences which obey uniform distribution over $[0\ 2]$. In this simulation example, we set the time horizon $T=161$, scalars $\gamma_1=0.8$, $\gamma_2=0.5$, $\gamma_3=0.01$, $\gamma_4=0.75$, $\gamma_5=0.1$, and positive definite matrices $W_1=W_2=9.875I$, and the initial values $x(0)$ is chosen as $x(0)=[2.8\ 1.6\ 4.6\ -0.2\ -0.8\ -3.8\ 0.1-0.2\ 2.8\ -2.9]$. Specify the reference formation matrix by $l(k)=[0.2\ 0\ 0.4\ 0\ 0.6\ 0\ 0.8\ 0\ 1\ 0]^T$. Assume that the probability $p_l^i(k)$ of agent i transmitting data to agent l at the discrete time k is shown in Table 1. In accordance with Algorithm 1, the observer and controller parameters $\left\{Q(k),K(k),\bar{K}(k)\right\}_{0\leqslant k\leqslant T-1}$ are respectively listed in Table 2.

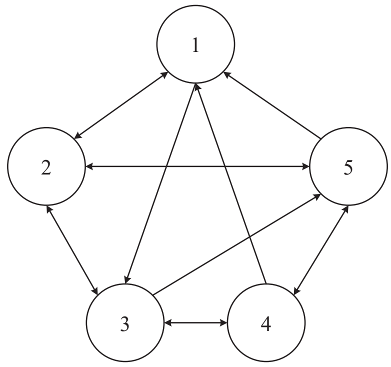

Figure 2. Directed communication graph G.

Table 1 The probability of transmitting data between agents

$p_l^i(k)$	$l=1$	$l=2$	$l=3$	$l=4$	$l=5$
$i=1$	0	0.4	0.6	0	0
$i=2$	0.5	0	0.5	0	0
$i=3$	0	0.4	0	0.3	0.3
$i=4$	0.4	0	0.3	0	0.3
$i=5$	0.2	0.3	0	0.5	0

Table 2 Observer and controller parameters $\left\{Q(k), K(k), \bar{K}(k)\right\}_{0\leqslant k\leqslant T-1}$

k	**1**	**2**	**3**	**4**	**5**	$\cdots$
$Q(k)$	$\begin{bmatrix}-0.0030\\-0.0026\\0.0013\end{bmatrix}$	$\begin{bmatrix}-0.0028\\-0.0022\\0.0011\end{bmatrix}$	$\begin{bmatrix}-0.0026\\-0.0019\\0.0009\end{bmatrix}$	$\begin{bmatrix}-0.0024\\-0.0018\\0.0009\end{bmatrix}$	$\begin{bmatrix}-0.0023\\-0.0018\\0.0010\end{bmatrix}$	$\cdots$
$K(k)$	$\begin{bmatrix}0.8635 & -0.2155\end{bmatrix}$	$\begin{bmatrix}0.7099 & -0.1780\end{bmatrix}$	$\begin{bmatrix}0.5806 & -0.1485\end{bmatrix}$	$\begin{bmatrix}0.4849 & -0.1281\end{bmatrix}$	$\begin{bmatrix}0.4253 & -0.1169\end{bmatrix}$	$\cdots$
$\bar{K}(k)$	$\begin{bmatrix}2.5000 & 2.5000\end{bmatrix}$	$\begin{bmatrix}2.4627 & 2.6915\end{bmatrix}$	$\begin{bmatrix}2.4713 & 2.8775\end{bmatrix}$	$\begin{bmatrix}2.4418 & 3.0364\end{bmatrix}$	$\begin{bmatrix}2.4145 & 3.1481\end{bmatrix}$	$\cdots$

The simulation results are presented in Figures 3-8, where the red dash dotted lines indicate the discrete time when faults occur, i.e., $k=85$. The state estimation errors $e_i^x(k), i=1,2,3,4,5$ are shown in Figure 3, where $e_i^{x,(1)}$ and $e_i^{x,(2)}$ represent two first-order components of $e_i^x(k)$, respectively. And Figure 4 depicts the faults $f_i(k), i=1,2,3,4,5$, and the fault estimations $\hat{f}_i(k), i=1,3$. In order to demonstrate the effectiveness of the proposed fault-tolerant formation consensus controller (11), we compare it with the formation consensus controller without the time-varying fault compensation term, i.e.,

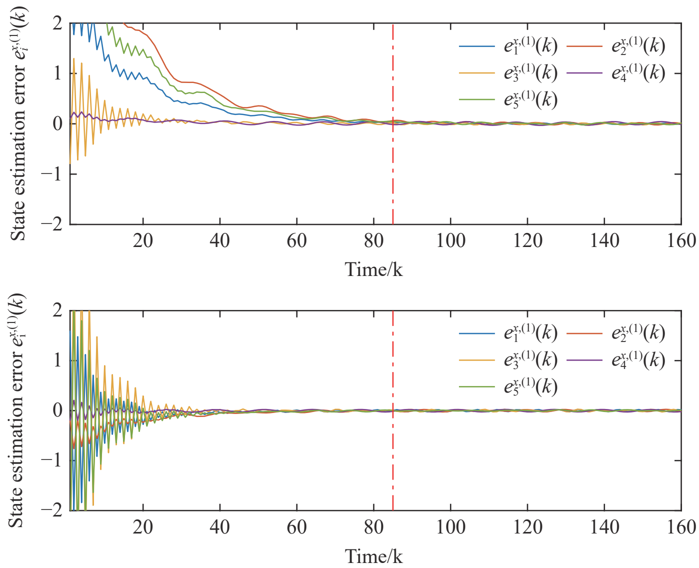

Figure 3. State estimation error $e_i^{x,(1)}(k)$ (upside) and $e_i^{x,(2)}(k)$ (downside), $i=1,2,3,4,5$.

$$\tilde{u}(k) \triangleq (I_N \otimes K(k)) \left(\vec{L}(\varepsilon(k)) \otimes I_{n_x}\right) \left(e^f(k) - \left(I_N \otimes \bar{E}\right) e^{\xi}(k)\right) - \left(I_N \otimes \bar{K}(k)\right) \tilde{l}(k),$$

and the state trajectories $x_i(k), i = 1,2,3,4,5$ of agents by using the developed controller $u(k)$ given by (11) and $\tilde{u}(k)$ are shown in Figures 5 and 6.

Obviously, when $k < 85$, both $u(k)$ and $\tilde{u}(k)$ can make MAS achieve the formation consensus control well. Moreover, the corresponding formation consensus errors $e_i^c(k), i = 1,2,3,4,5$, are depicted by Figures 7 and 8, which clearly reveal that, the proposed fault-tolerant controller (11), compared with the one without time-varying fault compensation term, can significantly suppress the effect of the faults on H_∞ formation consensus. It can be seen that the simulation results verify the validity and feasibility of the proposed fault-tolerant H_∞ control scheme.

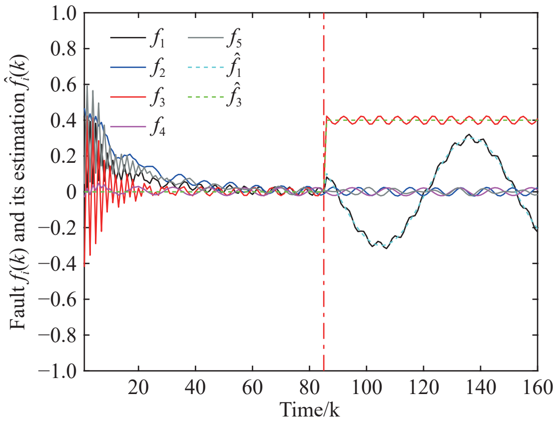

Figure 4. Fault $f_i(k), i = 1,2,3,4,5$, and its estimation $\hat{f}_i(k), i = 1,3$.

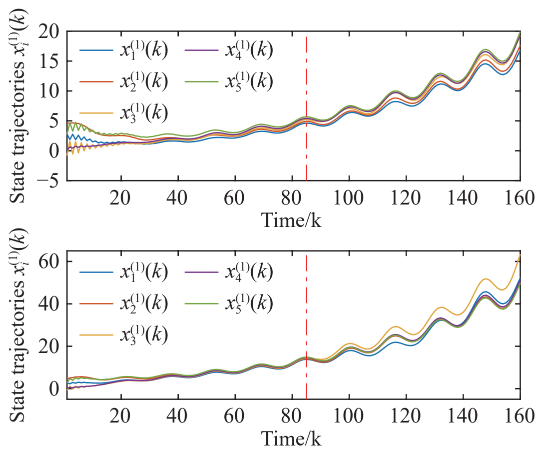

Figure 5. State trajectories $x_i^{(1)}(k), i = 1,2,3,4,5$, with $u(k)$ (upside) and $\tilde{u}(k)$ (downside).

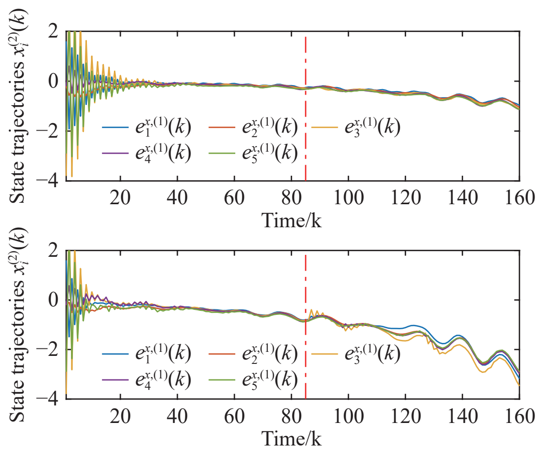

Figure 6. State trajectories $x_i^{(2)}(k), i = 1,2,3,4,5$, with $u(k)$ (upside) and $\tilde{u}(k)$ (downside).

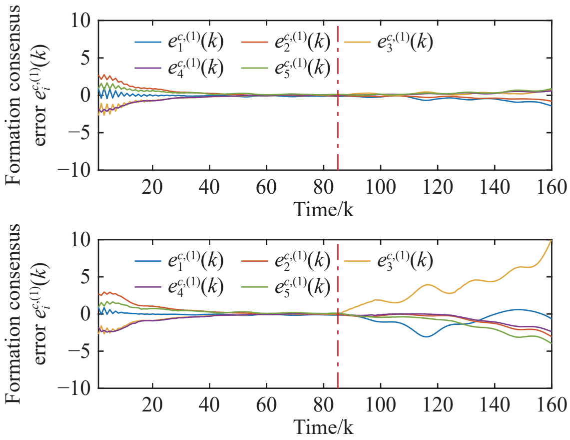

Figure 7. Formation consensus error $e_i^{c,(1)}(k), i = 1,2,3,4,5$, with $u(k)$ (upside) and $\tilde{u}(k)$ (downside).

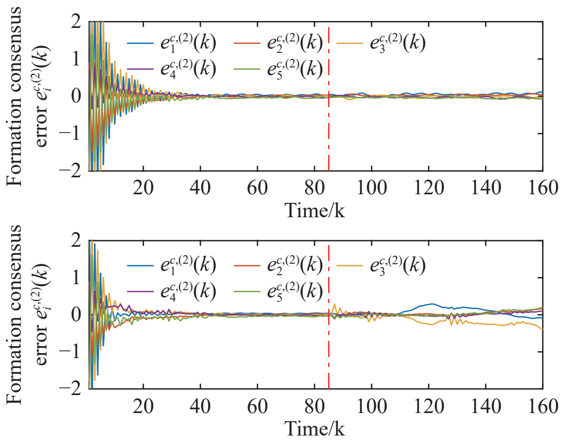

Figure 8. Formation consensus error $e_i^{c,(2)}(k), i = 1,2,3,4,5$, with $u(k)$ (upside) and $\tilde{u}(k)$ (downside).

5. Conclusions

In this paper, the problem of finite-horizon fault-tolerant distributed formation consensus control has been addressed for a LTV MAS with SCP. To this end, a singular system has been constructed by augmenting the state and fault of the system into a new vector, and a decentralized observer has been designed. A sufficient condition for the existence of observers has been given to ensure that the state estimation error system satisfies the given H_∞ performance constraint. By resorting to describing the scheduling behavior of the SCP by a stochastic variable sequence, the closed-loop MAS has been modeled as a time-varying system with stochastic parameter matrices. Utilizing the obtained state and fault information, fault-tolerant distributed formation consensus controllers have been constructed, which can suppress the adverse effects mainly induced by faults and external disturbances. In addition, a sufficient condition has been given to ensure that the MAS achieves formation consensus, and the parameters of the controller have been obtained by solving two coupled backward recursive RDEs. Finally, an illustrative example has been presented to validate the feasibility and effectiveness of the developed methods.

Author Contributions: Chunyu Li: Draft writing, conceptualization, methodology, simulation. **Yifan Liu:** Investigation, simulation. **Ming Gao:** Supervision, review & editing. **Li Sheng:** Funding acquisition, review & editing, supervision.

Funding: This work was supported in part by the National Natural Science Foundation of China under Grant 62073339, Grant 62173343, and Grant 62033008; in part by the Natural Science Foundation of Shandong Province of China under Grant ZR2022ZD34; and in part by the Research Fund for the Taishan Scholar Project of Shandong Province of China.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Cheng, P.D.C.; Indri, M.; Possieri, C.; *et al*. Path planning in formation and collision avoidance for multi-agent systems. Nonlinear Anal. Hybrid Syst., **2023**, *47*: 101293. doi: 10.1016/j.nahs.2022.101293
2. Xu, B.W.; Zhang, H.T.; Ding, Y.; *et al*. Event-triggered surrounding formation control of multiagent systems for multiple dynamic targets. IEEE Trans. Control Netw. Syst., **2023**, *10*: 752−764. doi: 10.1109/TCNS.2022.3210291
3. Su, Y.F.; Cai, H.; Huang, J. The cooperative output regulation by the distributed observer approach. Int. J. Netw. Dyn. Intell., **2022**, *1*: 20−35. doi: 10.53941/ijndi0101003
4. Yan, B.; Shi, P.; Lim, C.C.; *et al*. Robust formation control for multiagent systems based on adaptive observers. IEEE Syst. J., **2022**, *16*: 3139−3150. doi: 10.1109/JSYST.2021.3127579
5. Wu, J.F.; Wang, H.L.; Li, N.; *et al*. Formation obstacle avoidance: A fluid-based solution. IEEE Syst. J., **2020**, *14*: 1479−1490. doi: 10.1109/JSYST.2019.2917786
6. Liu, M.C.; Li, Y.D.; Zhu, L.; *et al*. Formation control for mixed-order UAVs-USVs-UUVs systems under cooperative and optimal control. J. Mar. Sci. Eng., **2023**, *11*: 704. doi: 10.3390/jmse11040704
7. Shi, P.; Yan, B. A survey on intelligent control for multiagent systems. IEEE Trans. Syst., Man, Cybern.: Syst., **2021**, *51*: 161−175. doi: 10.1109/TSMC.2020.3042823
8. Ma, L.F.; Wang, Z.D.; Chen, Y.; *et al*. Probability-guaranteed distributed filtering for nonlinear systems with innovation constraints over sensor networks. IEEE Trans. Control Netw. Syst., **2021**, *8*: 951−963. doi: 10.1109/TCNS.2021.3049361
9. Niu, Y.C.; Sheng, L.; Gao, M.; Zhou, D.H. Distributed intermittent fault detection for linear stochastic systems over sensor network. IEEE Trans. Cybern., **2022**, *52*: 9208−9218. doi: 10.1109/TCYB.2021.3054123
10. Luo, Y.Q.; Wang, Z.D.; Hu, J.; *et al*. Security-guaranteed fuzzy networked state estimation for 2-D systems with multiple sensor arrays subject to deception attacks. IEEE Trans. Fuzzy Syst., **2023**, *31*: 3624−3638. doi: 10.1109/TFUZZ.2023.3262609
11. Sheng, L.; Zhang, S.; Gao, M. Intermittent fault detection for linear discrete-time stochastic multi-agent systems. Appl. Math. Comput., **2021**, *410*: 126480. doi: 10.1016/j.amc.2021.126480
12. Shakiba, F.M.; Shojaee, M.; Azizi, S.M.; *et al*. Real-time sensing and fault diagnosis for transmission lines. Int. J. Netw. Dyn. Intell., **2022**, *1*: 36−47. doi: 10.53941/ijndi0101004
13. Amin, A.A.; Hasan, K.M. A review of fault tolerant control systems: Advancements and applications. Measurement, **2019**, *143*: 58−68. doi: 10.1016/j.measurement.2019.04.083
14. Sheng, L.; Niu, Y.C.; Gao, M.; *et al*. Particle filter-based fault detection for Toolface measurement of rotary steerable systems. IEEE Trans. Instrum. Meas., **2023**, *72*: 3519511. doi: 10.1109/TIM.2023.3284049
15. Hu, Q.; Xiao, B.; Friswell, M.I. Robust fault-tolerant control for spacecraft attitude stabilisation subject to input saturation. IET Control Theory Appl., **2011**, *5*: 271−282. doi: 10.1049/iet-cta.2009.0628
16. Wang, Q.; Dong, X.W.; Wang, B.H.; *et al*. Finite-time observer-based $H\infty$ fault-tolerant output formation tracking control for heterogeneous nonlinear multi-agent systems. IEEE Trans. Netw. Sci. Eng., **2023**, *10*: 1822−1834. doi: 10.1109/TNSE.2023.3235527
17. Nair, R.R.; Karki, H.; Shukla, A.; *et al*. Fault-tolerant formation control of nonholonomic robots using fast adaptive gain nonsingular terminal sliding mode control. IEEE Syst. J., **2019**, *13*: 1006−1017. doi: 10.1109/JSYST.2018.2794418
18. Deng, C.; Che, W.W. Fault-tolerant fuzzy formation control for a class of nonlinear multiagent systems under directed and switching topology. IEEE Trans. Syst., Man, Cybern.: Syst., **2021**, *51*: 5456−5465. doi: 10.1109/TSMC.2019.2954870
19. Liu, C.; Jiang, B.; Zhang, K. Adaptive fault-tolerant H-infinity output feedback control for Lead-Wing close formation flight. IEEE Trans. Syst., Man, Cybern.: Syst., **2020**, *50*: 2804−2814. doi: 10.1109/TSMC.2018.2830511
20. Liu, C.; Jiang, B.; Patton, R.J.; *et al*. Integrated fault-tolerant control for close formation flight. IEEE Trans. Aerosp. Electron. Syst., **2020**, *56*: 839−852. doi: 10.1109/TAES.2019.2920221
21. Ji, W.Y.; Pan, Y.N.; Zhao, M. Adaptive fault-tolerant optimized formation control for perturbed nonlinear multiagent systems. Int. J. Robust Nonlinear Control, **2022**, *32*: 3386−3407. doi: 10.1002/rnc.5961
22. Yu, Z.Q.; Zhang, Y.M.; Jiang, B.; *et al*. Distributed fractional-order intelligent adaptive fault-tolerant formation-containment control of two-layer networked unmanned airships for safe observation of a smart city. IEEE Trans. Cybern., **2022**, *52*: 9132−9144. doi: 10.1109/TCYB.2021.3052875
23. Liu, D.Y.; Liu, H.; Lewis, F.L.; *et al*. Robust fault-tolerant formation control for tail-sitters in aggressive flight mode transitions. IEEE Trans. Ind. Inf., **2020**, *16*: 299−308. doi: 10.1109/TII.2019.2914478
24. Wang, H.Q.; Dong, J.X. Time-varying formation control for multi-agent systems under directed topology base on gain re-adaptation fault-tolerant compensation approach. Int. J. Robust Nonlinear Control, **2022**, *32*: 3909−3922. doi: 10.1002/rnc.6007
25. Liu, Y.S.; Dong, X.W.; Shi, P.F.; *et al*. Distributed fault-tolerant formation tracking control for multiagent systems with multiple leaders and constrained actuators. IEEE Trans. Cybern., **2023**, *53*: 3738−3747. doi: 10.1109/TCYB.2022.3141734
26. Ren, Y.; Zhang, K.; Jiang, B.; *et al*. Distributed fault-tolerant time-varying formation control of heterogeneous multi-agent systems. Int. J. Robust Nonlinear Control, **2022**, *32*: 2864−2882. doi: 10.1002/rnc.5870
27. Jiang, J.H.; Jiang, Y.Y. Leader-following consensus of linear time-varying multi-agent systems under fixed and switching topologies. Automatica, **2020**, *113*: 108804. doi: 10.1016/j.automatica.2020.108804
28. Li, H.Y.; Li, X. Distributed model predictive consensus of heterogeneous time-varying multi-agent systems: With and without self-triggered mechanism. IEEE Trans. Circuits Syst. I:Regul. Pap., **2020**, *67*: 5358−5368. doi: 10.1109/TCSI.2020.3008528
29. Li, H.Y.; Li, X. Distributed consensus of heterogeneous linear time-varying systems on UAVS-USVs coordination. IEEE Trans. Circuits Syst. II:Express Briefs, **2020**, *67*: 1264−1268. doi: 10.1109/TCSII.2019.2928870
30. Chen, Y.Z.; Huang, X.L.; Zhan, J.Y. Gain-scheduled robust control for multi-agent linear parameter-varying systems with communication delays. Int. J. Robust Nonlinear Control, **2022**, *32*: 792−806. doi: 10.1002/rnc.5854
31. Cai, Y.L.; Zhang, H.G.; Zhang, K.; *et al*. Distributed leader-following consensus of heterogeneous second-order time-varying nonlinear multi-agent systems under directed switching topology. Neurocomputing, **2019**, *325*: 31−47. doi: 10.1016/j.neucom.2018.09.068
32. Chen, J.X.; Li, J.M.; Yang, N.N. Globally repetitive learning consensus control of unknown nonlinear multi-agent systems with uncertain time-varying parameters. Appl. Math. Modell., **2021**, *89*: 348−362. doi: 10.1016/j.apm.2020.07.063
33. Zeng, Z.X.; Peng, S.G.; Feng, W.D. Quasi-consensus of time-varying multi-agent systems with external inputs under deception attacks. Entropy, **2022**, *24*: 447. doi: 10.3390/e24040447
34. Chen, J.X.; Yu, Z.H.; Liu, S.Y.; *et al*. Fully distributed neural control of periodically time-varying parameterized stochastic nonlin-

ear multi-agent systems with hybrid-order dynamics. Appl. Math. Comput., **2022**, *426*: 127117. doi: 10.1016/j.amc.2022.127117
35. Liu, J.H.; Wang, C.L.; Xu, Y.J. Distributed adaptive output consensus tracking for high-order nonlinear time-varying multi-agent systems with output constraints and actuator faults. J. Franklin Inst., **2020**, *357*: 1090−1117. doi: 10.1016/j.jfranklin.2019.12.015
36. Liu, L.; Sun, H.; Ma, L.F.; *et al*. Quasi-consensus control for a class of time-varying stochastic nonlinear time-delay multiagent systems subject to deception attacks. IEEE Trans. Syst., Man, Cybern.: Syst., **2021**, *51*: 6863−6873. doi: 10.1109/TSMC.2020.2964826
37. Geng, H.; Wang, Z.D.; Chen, Y.; *et al*. Variance-constrained filtering fusion for nonlinear cyber-physical systems with the denial-of-service attacks and stochastic communication protocol. IEEE/CAA J. Autom. Sin., **2022**, *9*: 978−989. doi: 10.1109/JAS.2022.105623
38. Zou, L.; Wang, Z.D.; Hu, J.; *et al*. Communication-protocol-based analysis and synthesis of networked systems: Progress, prospects and challenges. Int. J. Syst. Sci., **2021**, *52*: 3013−3034. doi: 10.1080/00207721.2021.1917721
39. Zhao, Z.Y.; Wang, Z.D.; Zou, L.; *et al*. Zonotopic multi-sensor fusion estimation with mixed delays under try-once-discard protocol: A set-membership framework. Inf. Fusion, **2023**, *91*: 681−693. doi: 10.1016/j.inffus.2022.11.012
40. Luo, Y.Q.; Wang, Z.D.; Chen, Y.; *et al*. H_∞ state estimation for coupled stochastic complex networks with periodical communication protocol and intermittent nonlinearity switching. IEEE Trans. Netw. Sci. Eng., **2021**, *8*: 1414−1425. doi: 10.1109/TNSE.2021.3058220
41. Luo, Y.Q.; Wang, Z.D.; Sheng, W.G.; *et al*. State estimation for discrete time-delayed impulsive neural networks under communication constraints: A delay-range-dependent approach. IEEE Trans. Neural Netw. Learn. Syst., **2023**, *34*: 1489−1501. doi: 10.1109/TNNLS.2021.3105449
42. Wang, X.L.; Sun, Y.; Ding, D.R. Adaptive dynamic programming for networked control systems under communication constraints: A survey of trends and techniques. Int. J. Netw. Dyn. Intell., **2022**, *1*: 85−98. doi: 10.53941/ijndi0101008
43. Tabbara, M.; Nesic, D. Input-output stability of networked control systems with stochastic protocols and channels. IEEE Trans. Autom. Control, **2008**, *53*: 1160−1175. doi: 10.1109/TAC.2008.923691
44. Zhu, K.Q.; Hu, J.; Liu, Y.R.; *et al*. On ℓ_2-ℓ_∞ output-feedback control scheduled by stochastic communication protocol for two-dimensional switched systems. Int. J. Syst. Sci., **2021**, *52*: 2961−2976. doi: 10.1080/00207721.2021.1914768
45. Cheng, J.; Yan, H.C.; Park, J.H.; *et al*. Output-feedback control for fuzzy singularly perturbed systems: A nonhomogeneous stochastic communication protocol approach. IEEE Trans. Cybern., **2023**, *53*: 76−87. doi: 10.1109/TCYB.2021.3089612
46. Styan, G.P.H. Hadamard products and multivariate statistical analysis. Linear Algebra Appl., **1973**, *6*: 217−240. doi: 10.1016/0024-3795(73)90023-2
47. Penrose, R. On best approximate solutions of linear matrix equations. Math. Proc. Cambridge Philos. Soc., **1956**, *52*: 17−19. doi: 10.1017/S0305004100030929

Citation: Li, C.; Liu, Y.; Gao, M.; *et al*. Fault-tolerant formation consensus control for time-varying multi-agent systems with stochastic communication protocol. *International Journal of Network Dynamics and Intelligence*. 2024, 3(1), 100004. doi: 10.53941/ijndi.2024.100004